2. a. $x(t)$ is sampled at a period T to form the DT signal $y(n)$.

Let $X(f) = \text{CTFT}\{x(t)\}$, then

$$\boxed{Y(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right)}$$

b. $Z(e^{j\omega}) = Y(e^{j\omega})H(e^{j\omega})$

$$\boxed{Z(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{2\pi T}\right) H\left(e^{j\omega}\right)}$$

c. $$r(t) = \sum_{m=-\infty}^{\infty} z(mT)\,\text{rect}\left(\frac{t - T/2 - mT}{T}\right)$$

$$= \left(\sum_{m=-\infty}^{\infty} z(mT)\,\delta(t - mT)\right) * \text{rect}\left(\frac{t - T/2}{T}\right)$$

$$R(f) = Z(2\pi Tf)\left(T\,\text{sinc}(Tf)\,e^{-j2\pi f(T/2)}\right)$$

$$R(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{(2\pi Tf) - 2\pi k}{2\pi T}\right)\left(H\left(e^{j\omega}\right)\Big|_{\omega = 2\pi Tf}\right)\left(T\,\text{sinc}(Tf)\,e^{-j\pi Tf}\right)$$

$$\boxed{R(f) = \sum_{k=-\infty}^{\infty} X\left(f - \frac{k}{T}\right) H\left(e^{j(2\pi Tf)}\right) \text{sinc}(Tf)\,e^{-j\pi Tf}}$$

d. $X_r(f) = R(f)\,\text{rect}(Tf)$

$$\boxed{X_r(f) = X(f)\;H(e^{j(2\pi Tf)})\,\text{sinc}(Tf)\,e^{-j\pi Tf}\,\text{rect}(Tf)}$$

2. To determine $H(e^{j\omega})$, we know that

$$H(e^{j(2\pi Tf)})\,\mathrm{sinc}(Tf)e^{-j\pi Tf}\,\mathrm{rect}(Tf) = \mathrm{rect}(Tf)$$

$$\Rightarrow H(e^{j(2\pi Tf)})\,\mathrm{rect}(Tf) = \frac{e^{j\pi Tf}}{\mathrm{sinc}(Tf)}\,\mathrm{rect}(Tf)$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{e^{j\pi T\left(\frac{\omega-2\pi k}{2\pi T}\right)}}{\mathrm{sinc}\left(T\left(\frac{\omega-2\pi k}{2\pi T}\right)\right)}\,\mathrm{rect}\left(T\left(\frac{\omega-2\pi k}{2\pi T}\right)\right)$$

$$\boxed{H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{e^{j\left(\frac{\omega-2\pi k}{2}\right)}}{\mathrm{sinc}\left(\frac{\omega-2\pi k}{2\pi}\right)}\,\mathrm{rect}\left(\frac{\omega-2\pi k}{2\pi}\right)}$$

3. $H(e^{j\omega}) = 2\,\mathrm{rect}\left(\frac{\omega}{\pi}\right)$ for $|\omega| < \pi$

$$y(n) = \begin{cases} x\left(\frac{n}{2}\right) & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases}$$

$z(n) = y(n) * h(n)$ where $h(n) = \mathrm{IDTFT}\{H(e^{j\omega})\}$

$$z(n) = \sum_{K=-\infty}^{\infty} y(K)\,h(n-K)$$

$$Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \sum_{K=-\infty}^{\infty} y(K)\,h(n-K)\,e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{\substack{K=-\infty \\ K \text{ even}}}^{\infty} x\left(\tfrac{K}{2}\right) h(n-K)\,e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x(l)\,h(n-2l)\,e^{-j\omega n}$$

$$= \sum_{l=-\infty}^{\infty} x(l) \sum_{n=-\infty}^{\infty} h(n-2l)\,e^{-j\omega n}$$

$$= \sum_{l=-\infty}^{\infty} x(l)\,e^{-j2l\omega}\,H(e^{j\omega})$$

$$= H(e^{j\omega}) \sum_{l=-\infty}^{\infty} x(l)\,e^{-jl(2\omega)}$$

$$= H(e^{j\omega})\,X(2\omega)$$

3. a. $x(n) = \delta(n)$

$$X(e^{j\omega}) = 1$$

$$X(e^{j(2\omega)}) = 1$$

$$Z(e^{j\omega}) = 2\,\mathrm{rect}\left(\frac{\omega}{\pi}\right)(1) \qquad \text{for } \omega \in (-\pi, \pi)$$

$$z(n) = \frac{2\sin\left(\frac{\pi}{2}n\right)}{\pi n} = \frac{2\sin\left(\pi\frac{n}{2}\right)}{2\left(\pi\frac{n}{2}\right)}$$

$$\boxed{z(n) = \mathrm{sinc}\left(\frac{n}{2}\right)}$$

b. $x(n) = \delta(n-1)$

$$X(e^{j\omega}) = e^{-j\omega}(1) \qquad \text{for } \omega \in (-\pi, \pi)$$

$$X(e^{j(2\omega)}) = e^{-j2\omega}(1) \qquad \text{for } \omega \in (-\pi, \pi)$$

$$Z(e^{j\omega}) = X(e^{j(2\omega)})H(e^{j\omega})$$

$$= e^{-j2\omega}\left(2\,\mathrm{rect}\left(\frac{\omega}{\pi}\right)\right) \qquad \text{for } \omega \in (-\pi, \pi)$$

$$\boxed{z(n) = \mathrm{sinc}\left(\frac{n-2}{2}\right)}$$

c. $x(n) = 1$

$$X(e^{j\omega}) = 2\pi\delta(\omega) \quad \text{for } \omega \in [-\pi, \pi]$$

$$X(e^{j2\omega}) = 2\pi\delta(2\omega) = \pi\delta(\omega) \qquad \text{for } \omega \in (-\pi, \pi)$$

$$Z(e^{j\omega}) = X(e^{j2\omega})H(e^{j\omega})$$

$$= \pi\delta(\omega)\left(2\,\mathrm{rect}\left(\frac{\omega}{\pi}\right)\right) \quad \text{for } \omega \in (-\pi, \pi)$$

$$= 2\pi\delta(\omega) \qquad \text{for } \omega \in (-\pi, \pi)$$

$$\boxed{z(n) = 1}$$

3. d. $x(n) = \cos\left(\frac{\pi n}{4}\right)$

$$X(e^{j\omega}) = \pi\left[\delta(\omega - \tfrac{\pi}{4}) + \delta(\omega + \tfrac{\pi}{4})\right] \quad ; \; |\omega| < \pi$$

$$X(e^{j2\omega}) = \pi\left[\delta(2\omega - \tfrac{\pi}{4}) + \delta(2\omega + \tfrac{\pi}{4})\right] \quad ; \; |\omega| < \pi$$

$$= \frac{\pi}{2}\left[\delta(\omega - \tfrac{\pi}{8}) + \delta(\omega + \tfrac{\pi}{8})\right] \quad ; \; |\omega| < \pi$$

$$Z(e^{j\omega}) = X(e^{j2\omega})\,H(e^{j\omega})$$

$$= \frac{\pi}{2}\left[\delta(\omega - \tfrac{\pi}{8}) + \delta(\omega + \tfrac{\pi}{8})\right] 2\,\mathrm{rect}\left(\frac{\omega}{\pi}\right) ; \quad |\omega| < \pi$$

$$= \pi\left[\delta(\omega - \tfrac{\pi}{8}) + \delta(\omega + \tfrac{\pi}{8})\right] \quad ; \; |\omega| < \pi$$

$$\boxed{z(n) = \cos\left(\frac{\pi}{8} n\right)}$$

e. $x(n) = \mathrm{sinc}\left(\frac{n}{8}\right) = \dfrac{\sin\left(\pi \frac{n}{8}\right)}{\pi \frac{n}{8}} = \dfrac{8 \sin\left(\frac{\pi}{8} n\right)}{\pi n}$

$$X(e^{j\omega}) = 8\,\mathrm{rect}\left(\frac{\omega}{\frac{\pi}{4}}\right) = 8\,\mathrm{rect}\left(\frac{4\omega}{\pi}\right) \quad ; \; |\omega| < \pi$$

$$X(e^{j2\omega}) = 8\,\mathrm{rect}\left(\frac{8\omega}{\pi}\right) \quad ; \; |\omega| < \pi$$

$$Z(e^{j2\omega}) = X(e^{j2\omega})\,H(e^{j\omega})$$

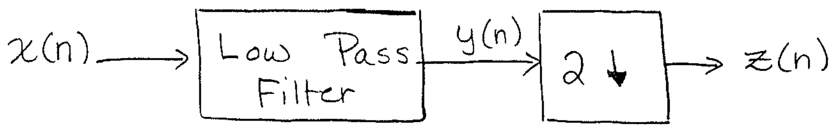
$$= 8\,\mathrm{rect}\left(\frac{8\omega}{\pi}\right) 2\,\mathrm{rect}\left(\frac{\omega}{\pi}\right) \quad ; \; |\omega| < \pi$$

$$= 16\,\mathrm{rect}\left(\frac{8\omega}{\pi}\right) \quad ; \; |\omega| < \pi$$

$$z(n) = \frac{16 \sin\left(\frac{\pi}{16} n\right)}{\pi n} = \frac{\sin\left(\frac{\pi}{16} n\right)}{\frac{\pi}{16} n} = \mathrm{sinc}\left(\frac{n}{16}\right)$$

$$\boxed{z(n) = \mathrm{sinc}\left(\frac{n}{16}\right)}$$

4. Compute $z(n)$ when we have the following system.



$$z(n) = y(2n)$$

$$
\begin{aligned}
Z(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} z(n) e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} y(2n) e^{-j\omega n} \\
&= \sum_{l=-\infty}^{\infty} \frac{(1+(-1)^l)}{2} y(l) e^{-j\frac{\omega}{2}l} \\
&= \frac{1}{2} \sum_{l=-\infty}^{\infty} \left(y(l) e^{-j\frac{\omega}{2}l} + y(l) e^{-j(\frac{\omega}{2}-\pi)l} \right) \\
&= \frac{1}{2} \left(Y(e^{j\frac{\omega}{2}}) + Y(e^{j(\frac{\omega}{2}-\pi)}) \right)
\end{aligned}
$$

a. $Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})$

$x(n) = \delta(n) \Rightarrow X(e^{j\omega}) = 1 \quad |\omega| < \pi$

and $Y(e^{j\omega}) = H(e^{j\omega}) = \text{rect}\left(\frac{\omega}{\pi}\right) \quad |\omega| < \pi$

$$y(n) = \frac{\sin \frac{\pi}{2} n}{\pi n} = \frac{1}{2} \frac{\sin\left(\pi \frac{n}{2}\right)}{\pi \left(\frac{n}{2}\right)} = \frac{1}{2} \text{sinc}\left(\frac{n}{2}\right)$$

$$z(n) = y(2n) = \frac{1}{2} \text{sinc}\left(\frac{2n}{2}\right)$$

$$\boxed{z(n) = \frac{1}{2} \text{sinc}(n)}$$

4. b. $x(n) = \delta(n-1)$

$X(e^{j\omega}) = e^{-j\omega} \quad ; |\omega| < \pi$

$Y(e^{j\omega}) = e^{-j\omega}\, \text{rect}\left(\frac{\omega}{\pi}\right) ; |\omega| < \pi$

$\Rightarrow y(n) = \frac{1}{2}\, \text{sinc}\left(\frac{n-1}{2}\right)$

$z(n) = y(2n) = \frac{1}{2}\, \text{sinc}\left(\frac{(2n)-1}{2}\right)$

$$\boxed{z(n) = \frac{1}{2}\, \text{sinc}\left(n - \frac{1}{2}\right)}$$

c. $x(n) = 1 \implies X(e^{j\omega}) = 2\pi\, \delta(\omega) \quad ; |\omega| < \pi$

$Y(e^{j\omega}) = 2\pi\, \delta(\omega)\, \text{rect}\left(\frac{\omega}{\pi}\right) = 2\pi\delta(\omega) \quad ; |\omega| < \pi$

$\Rightarrow y(n) = 1$

and $z(n) = y(2n) = 1$

$\boxed{z(n) = 1}$

d. $x(n) = \cos\left(\frac{\pi}{4}n\right) \Rightarrow X(e^{j\omega}) = \pi\left[\delta(\omega + \frac{\pi}{4}) + \delta(\omega - \frac{\pi}{4})\right] ; |\omega| < \pi$

$Y(e^{j\omega}) = \pi\left[\delta(\omega+\frac{\pi}{4}) + \delta(\omega - \frac{\pi}{4})\right] \text{rect}\left(\frac{\omega}{\pi}\right) ; |\omega| < \pi$

$= \pi\left[\delta(\omega + \frac{\pi}{4}) + \delta(\omega - \frac{\pi}{4})\right]$

$y(n) = \cos\left(\frac{\pi}{4}n\right)$

$z(n) = y(2n)$

$z(n) = \cos\left(\frac{\pi}{4}(2n)\right)$

$$\boxed{z(n) = \cos\left(\frac{\pi}{2}n\right)}$$

4. e. $x(n) = \text{sinc}\left(\frac{n}{8}\right) \Rightarrow X(e^{j\omega}) = 8\,\text{rect}\left(\frac{\omega}{(\frac{\pi}{4})}\right) \quad ; |\omega| < \pi$

$$Y(e^{j\omega}) = 8\,\text{rect}\left(\frac{\omega}{(\frac{\pi}{4})}\right)\text{rect}\left(\frac{\omega}{\pi}\right) \quad ; \quad |\omega| < \pi$$

$$= 8\,\text{rect}\left(\frac{\omega}{(\frac{\pi}{4})}\right) ; \quad |\omega| < \pi$$

$$\Rightarrow y(n) = \text{sinc}\left(\frac{n}{8}\right)$$

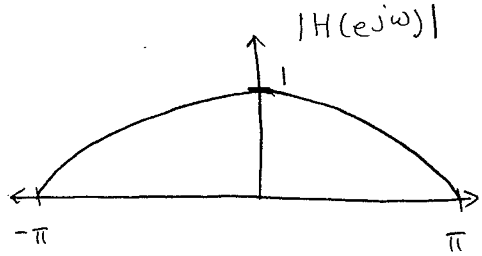
$$z(n) = y(2n) = \text{sinc}\left(\frac{2n}{8}\right)$$

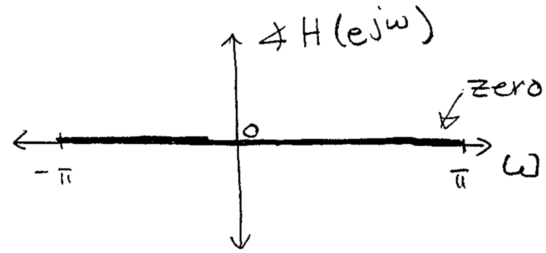
$$\boxed{z(n) = \text{sinc}\left(\frac{n}{4}\right)}$$

5. a) $y(n) = 0.25(x(n+1) + 2x(n) + x(n-1))$

$$\Rightarrow h(n) = 0.25(\delta(n+1) + 2\delta(n) + \delta(n-1))$$
$$\Rightarrow H(e^{j\omega}) = 0.25(e^{j\omega} + 2 + e^{-j\omega})$$
$$= 0.25(e^{j\omega} + 1 + 1 + e^{-j\omega})$$
$$= 0.5\left(e^{j\frac{\omega}{2}}\cos\left(\tfrac{\omega}{2}\right) + e^{-j\frac{\omega}{2}}\cos\left(\tfrac{\omega}{2}\right)\right)$$
$$= 0.5\left(e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}\right)\cos\left(\tfrac{\omega}{2}\right)$$
$$\boxed{H(e^{j\omega}) = \cos^2\left(\tfrac{\omega}{2}\right)} \qquad |\omega| < \pi$$

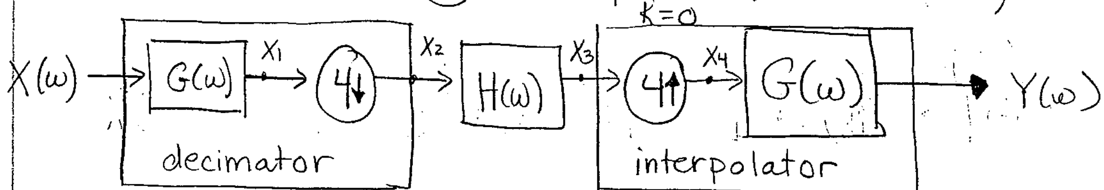
b)





c) $X(e^{j\omega}) \rightarrow (4\uparrow) \rightarrow X(e^{j4\omega})$

and $X(e^{j\omega}) \rightarrow (4\downarrow) \rightarrow \frac{1}{4}\sum_{k=0}^{3} X\left(e^{j(\omega+2\pi k)/4}\right)$

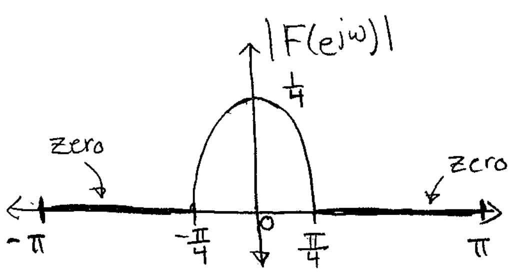


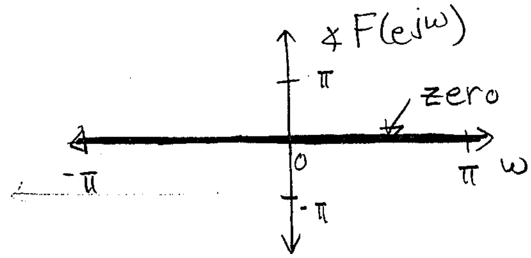
$$X_1(\omega) = X(\omega)\,\mathrm{rect}\left(\frac{\omega}{\frac{\pi}{2}}\right)$$

$$X_2(\omega) = \frac{1}{4}\sum_{k=0}^{3} X_1\left(\frac{\omega+2\pi k}{4}\right)$$
$$= \frac{1}{4}\sum_{k=0}^{3} X\left(\frac{\omega+2\pi k}{4}\right)\mathrm{rect}\left(\frac{\frac{\omega+2\pi k}{4}}{\frac{\pi}{2}}\right)$$
$$= \frac{1}{4}\sum_{k=0}^{3} X\left(\frac{\omega+2\pi k}{4}\right)\mathrm{rect}\left(\frac{\omega+2\pi k}{2\pi}\right)$$

5. c)
$$X_3(\omega) = X_2(\omega) H(\omega)$$
$$= X_2(\omega)\cos^2\left(\frac{\omega}{2}\right)$$
$$= \left(\frac{1}{4}\sum_{k=0}^{3} X\left(\frac{\omega + 2\pi k}{4}\right)\text{rect}\left(\frac{\omega + 2\pi k}{2\pi}\right)\right)\cos^2\left(\frac{\omega}{2}\right)$$
$$X_4(\omega) = X_3(4\omega)$$
$$X_4(\omega) = \left(\frac{1}{4}\sum_{k=0}^{3} X\left(\frac{4\omega + 2\pi k}{4}\right)\text{rect}\left(\frac{4\omega + 2\pi k}{2\pi}\right)\right)\cos^2\left(\frac{4\omega}{2}\right)$$
$$= \left(\frac{1}{4}\sum_{k=0}^{3} X\left(\omega + \frac{\pi}{2}k\right)\text{rect}\left(\frac{\omega + \frac{\pi}{2}k}{\frac{\pi}{2}}\right)\right)\cos^2(2\omega)$$
$$Y(\omega) = X_4(\omega)\,\text{rect}\left(\frac{\omega}{\frac{\pi}{2}}\right)$$
$$Y(\omega) = \left(\frac{1}{4}\sum_{k=0}^{3} X\left(\omega + \frac{\pi}{2}k\right)\text{rect}\left(\frac{\omega + \frac{\pi}{2}k}{\frac{\pi}{2}}\right)\right)\cos^2(2\omega)\,\text{rect}\left(\frac{\omega}{\frac{\pi}{2}}\right)$$
$$= \frac{1}{4}X(\omega)\cos^2(2\omega)\,\text{rect}\left(\frac{\omega}{\frac{\pi}{2}}\right)$$
$$\boxed{F(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{4}\cos^2(2\omega)\,\text{rect}\left(\frac{\omega}{\frac{\pi}{2}}\right)}$$

d)





e) To implement the filter above, we would need $f(n) = \frac{1}{4}\text{sinc}\left(\frac{n}{4}\right) * \frac{1}{4}\left(\delta(n+4) + 2\delta(n) + \delta(n-4)\right)$ which is an IIR filter. Thus, it is impossible to implement this filter exactly and would require a large window to approximate. Whereas if the system is implemented as a decimator followed by the filter $H(\omega)$ followed by an interpolator, these are relatively standard operations to perform on sampled signals.