ECE 438 HW #4 Fall 2007

1. DTFT$\{x[n]\} = ?$, $|a| < 1$

a. $$x(n) = \int_{-\pi}^{\pi} \left(\frac{\cos(\omega)^3}{1 + |\omega|^{3/2}} \right) e^{j\omega n} \, d\omega$$

$$= 2\pi \times \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos(\omega)^3}{1 + |\omega|^{3/2}} e^{j\omega n} \, d\omega$$

IDTFT $$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \, d\omega$$

$$\Rightarrow \boxed{X(\omega) = 2\pi \frac{\cos(\omega)^3}{1 + |\omega|^{3/2}}}$$

b. $x[n] = a^n u[n]$

$$X(\omega) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} (a e^{-j\omega})^n u[n]$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n = \boxed{\frac{1}{1 - a e^{-j\omega}}}$$

c. $x[n] = a^n e^{j\omega_0 n} u[n]$

$$X(\omega) = \sum_{n=-\infty}^{\infty} a^n e^{j\omega_0 n} u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (a e^{-j(\omega - \omega_0)})^n$$

$$\boxed{X(\omega) = \frac{1}{1 - a e^{-j(\omega - \omega_0)}}}$$

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1.1 d. $x[n] = u[n] - u[n-N]$

$$X(\omega) = \sum_{n=-\infty}^{\infty} (u[n] - u[n-N]) e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} (e^{-j\omega})^n$$

$$= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} = \frac{e^{-j\omega N/2}(e^{j\omega N/2} - e^{-j\omega N/2})}{e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2})}$$

$$\boxed{X(\omega) = e^{-j\omega(N-1)/2} \frac{\sin(\frac{\omega N}{2})}{\sin(\frac{\omega}{2})}}$$

2. Let $T[\ ]$ be defined such that

$$y[n] = T[x[n]] \Rightarrow h[n] = T[\delta[n]]$$

$$x[n] = x[n] * \delta[n]$$

$$\Rightarrow y[n] = T[x[n] * \delta[n]]$$

$$= T\left[\sum_{K=-\infty}^{\infty} x[K]\, \delta[n-K]\right]$$

$$= \sum_{K=-\infty}^{\infty} x[K]\, T[\delta[n-K]] \quad \text{by Linearity}$$

$$= \sum_{K=-\infty}^{\infty} x[K]\, h[n-K] \quad \text{by Time Invarian}$$

$$= x[n] * h[n]$$

Thus, if a system is LTI then
$y[n] = \sum_{K=-\infty}^{\infty} x[K] h[n-K]$. Q.E.D.

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3. $y[n] = x[n] * h[n]$

Let $Y(\omega) = \text{DTFT}\{y[n]\}$

$X(\omega) = \text{DTFT}\{x[n]\}$

$H(\omega) = \text{DTFT}\{h[n]\}$

Then,

$$Y(\omega) = X(\omega)H(\omega)$$

and if $x[n] = e^{j\omega_0 n}$, then $X(\omega) = \delta(\omega-\omega_0)$

$\Rightarrow$

$$Y(\omega) = \delta(\omega-\omega_0)H(\omega)$$

$$\Rightarrow y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(\omega)e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)e^{j\omega n}\delta(\omega-\omega_0)\,d\omega$$

$\therefore$ $y[n] = \frac{1}{2\pi}H(\omega_0)e^{j\omega_0 n}$ Q.E.D.

and $\boxed{C = \frac{1}{2\pi}H(\omega_0)}$

4. $y[n] = ?$, when $H(e^{j\omega}) = \text{DTFT}\{h[n]\}$ and $x[n] =$

a. $x[n] = e^{j\omega_0 n}$

from problem 3,

$$\boxed{y[n] = \frac{1}{2\pi}H(e^{j\omega_0})e^{j\omega_0 n}}$$

b. $x[n] = e^{j(\omega_0 n + \phi)}$

$$\boxed{y[n] = \frac{1}{2\pi}H(e^{j\omega_0})e^{j(\omega_0 n+\phi)}}$$ since $e^{j\phi}$ is constant

4. c. $x[n] = \cos(\omega_o n) = \frac{1}{2}\left(e^{j\omega_o n} + e^{-j\omega_o n}\right)$

$$\boxed{y[n] = \frac{1}{4\pi}\left(H(e^{j\omega_o})e^{j\omega_o n} + H(e^{-j\omega_o})e^{-j\omega_o n}\right)}$$

d. $x[n] = \cos(\omega_o n + \phi) = \frac{1}{2}\left(e^{j(\omega_o n+\phi)} + e^{-j(\omega_o n+\phi)}\right)$

$$\boxed{y[n] = \frac{1}{4\pi}\left(H(e^{j\omega_o})e^{j(\omega_o n+\phi)} + H(e^{-j\omega_o})e^{-j(\omega_o n+\phi)}\right)}$$

e. $x[n] = \sin(\omega_o n) = \frac{1}{j2}\left(e^{j\omega_o n} - e^{-j\omega_o n}\right)$

$$\boxed{y[n] = \frac{1}{j4\pi}\left(H(e^{j\omega_o})e^{j\omega_o n} - H(e^{-j\omega_o})e^{-j\omega_o n}\right)}$$

f. $x[n] = \sin(\omega_o n + \phi) = \frac{1}{j2}\left(e^{j(\omega_o n+\phi)} - e^{-j(\omega_o n+\phi)}\right)$

$$\boxed{y[n] = \frac{1}{j4\pi}\left(H(e^{j\omega_o})e^{j(\omega_o n+\phi)} - H(e^{-j\omega_o})e^{-j(\omega_o n+\phi)}\right)}$$

5. $y[n] = x[n] + 0.5y[n-1]$

a) $h[n] = \delta[n] + 0.5h[n-1]$

$h[n] = 0$ for all $n<0$, then

$$h[0] = \delta[0] + 0.5h[0-1]$$
$$= 1 + 0 \qquad = 1$$
$$h[1] = \delta[1] + 0.5h[1-1]$$
$$= 0 + 0.5(1) \qquad = 0.5(1)$$
$$h[2] = 0 + 0.5h[1]$$
$$= 0 + 0.5(0.5(1)) = 0.5^2(1)$$

$\vdots$

$$\boxed{h[n] = 0.5^n u[n]}$$

b) $y[n] = x[n] + 0.5y[n-1]$

$$\text{DTFT}\{y[n]\} = \text{DTFT}\{x[n] + 0.5y[n-1]\}$$
$$Y(\omega) = X(\omega) + 0.5e^{-j\omega}Y(\omega)$$
$$Y(\omega)(1-0.5e^{-j\omega}) = X(\omega)$$

$$\boxed{H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1-0.5e^{-j\omega}}}$$

c) $$\sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} 0.5^n u[n]e^{-j\omega n}$$
$$= \sum_{n=0}^{\infty} (0.5e^{-j\omega})^n$$

$$\boxed{H(\omega) = \frac{1}{1-0.5e^{-j\omega}}}$$

Thus, the two methods result in the same answer.

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6. a. $y[n] = (x[n] + x[n-1])/2$

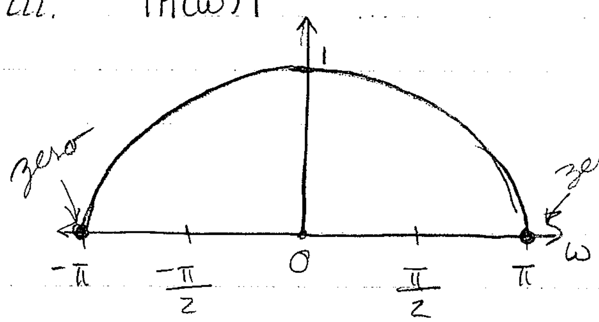
i. $\boxed{h[n] = (\delta[n] + \delta[n-1])/2}$

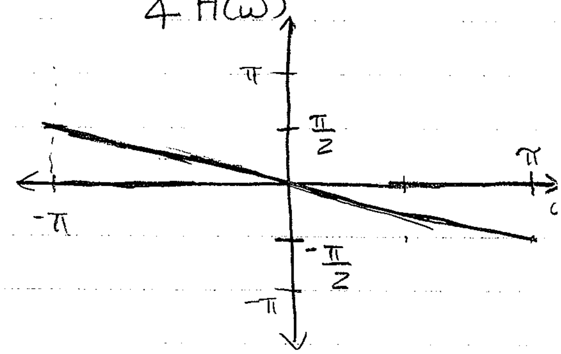
ii. $H(\omega) = (1 + e^{-j\omega})/2$

$= \frac{1}{2} e^{-j\omega/2} \left(e^{j\omega/2} + e^{-j\omega/2}\right)$

$\boxed{H(\omega) = e^{-j\omega/2} \cos(\omega/2)}$

iii. $|H(\omega)|$ $\measuredangle H(\omega)$





iv. The filter is a lowpass filter. So higher frequencies will be filtered out, and because of the phase response there will be some distortion (delay) at higher frequencies as compared to lower frequencies.

6.b. $y[n] = (x[n] + x[n-2])/2$

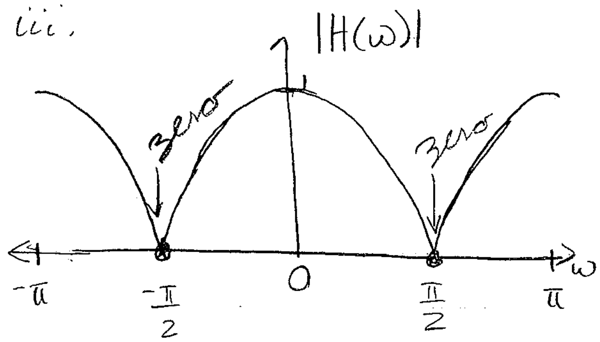
i. $\boxed{h[n] = (\delta[n] + \delta[n-2])/2}$

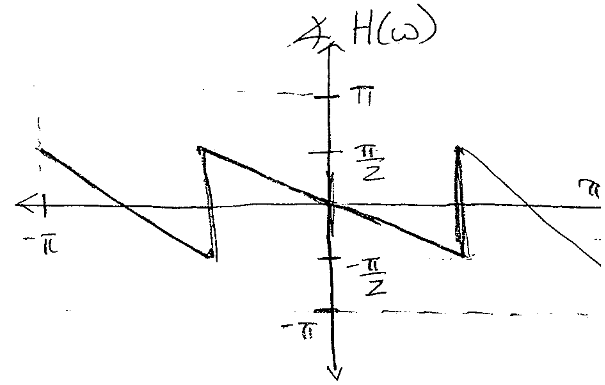
ii. $H(\omega) = (1 + e^{-j2\omega})/2$

$= \frac{1}{2} e^{-j\omega}(e^{j\omega} + e^{-j\omega})$

$\boxed{H(\omega) = e^{-j\omega}\cos(\omega)}$

iii.





iv. The filter is a bandstop filter. Frequency content at $\omega = \pm\frac{\pi}{2}$ will be filter out of the output. Also due to the phase response, there will be some distortion due to the difference in phase betweer frequencies.

6.c. $y[n] = x[n] + x[n-1] + y[n-1]$

i. $h[n] = \begin{cases} 0 & \text{for } n<0 \\ \delta[n] + \delta[n-1] + h[n-1] & \text{otherwise} \end{cases}$

$h[0] = 1 + 0 + 0 = 1$

$h[1] = 0 + 1 + 1 = 2$

$h[2] = 0 + 0 + 2 = 2$

$h[3] = 0 + 0 + 2 = 2$

$\vdots$

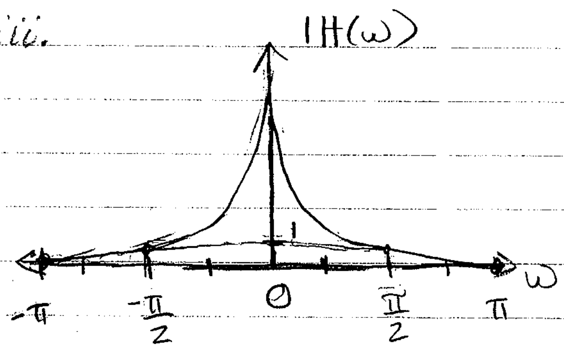
$$\boxed{h[n] = u[n] + u[n-1]}$$

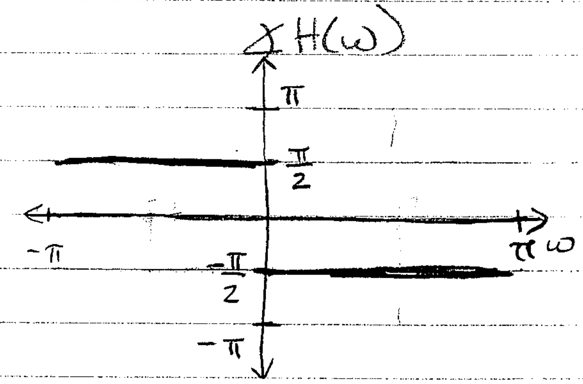
ii. $$H(\omega) = \frac{1}{1-e^{-j\omega}} + \frac{e^{-j\omega}}{1-e^{-j\omega}}$$

$$= \frac{2e^{-j\omega/2}\left(\frac{1}{2}\right)\left(e^{j\omega/2} + e^{-j\omega/2}\right)}{j\,2\,e^{-j\omega/2}\left(\frac{1}{j2}\right)\left(e^{j\omega/2} - e^{-j\omega/2}\right)}$$

$$= \frac{e^{-j\frac{\pi}{2}}\cos(\omega/2)}{\sin(\omega/2)}$$

iii.





iv. This filter is a lowpass filter with gain higher than 1 for any frequency lower than $\omega = \frac{\pi}{2}$. In particular low frequencie are amplified while high frequencies are annihilated.

7. a) $y[n] = x[n] * h_0[n] - x[n-1] * h_0[n]$

$$h[n] = \delta[n] * h_0[n] - \delta[n-1] * h_0[n]$$
$$\boxed{h[n] = h_0[n] - h_0[n-1]}$$

b) $\boxed{H(e^{j\omega}) = H_0(e^{j\omega}) - e^{-j\omega} H_0(e^{j\omega})}$

c) $y[n] = (x[n] + x[n-1])/2$

$$\boxed{h_0[n] = (\delta[n] + \delta[n-1])/2}$$

d) $\boxed{H_0(\omega) = e^{-j\omega/2} \cos(\omega/2)}$ See prob. 6 for work

e) $h[n] = h_0[n] - h_0[n-1]$
$$= (\delta[n] + \delta[n-1])/2 - (\delta[n-1] + \delta[n-2])/2$$
$$\boxed{h[n] = (\delta[n] - \delta[n-2])/2}$$

f) $H(\omega) = (1 - e^{-j2\omega})(\tfrac{1}{2})$
$$= j\frac{1}{j2} e^{-j\omega}(e^{j\omega} - e^{-j\omega})$$
$$\boxed{H(\omega) = e^{-j(\omega - \frac{\pi}{2})} \sin(\omega)}$$

8. a) $x[n] = u[n] - u[n-4]$; $h[n] = a^n u[n]$

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= (u[n] - u[n-4]) * h[n] \\ &= u[n] * h[n] - u[n-4] * h[n] \\ &= u[n] * h[n] - \delta[n-4] * u[n] * h[n] \end{aligned}$$

$$\begin{aligned} u[n] * h[n] &= \sum_{K=-\infty}^{\infty} a^K u[K]\, u[n-K] \\ &= \left(\sum_{K=0}^{n} a^K\right) u[n] \\ &= \frac{1 - a^{n+1}}{1-a}\, u[n] \end{aligned}$$

$$\begin{aligned} y[n] &= \frac{1-a^{n+1}}{1-a} u[n] - \delta[n-4] * \left(\frac{1-a^{n+1}}{1-a}\right) u[n] \\ &= \frac{1-a^{n+1}}{1-a} u[n] - \frac{1-a^{n-3}}{1-a} u[n-4] \end{aligned}$$

$$\boxed{y[n] = \frac{1}{1-a}\left[\left(1-a^{n+1}\right)u[n] - \left(1-a^{n-3}\right)u[n-4]\right]}$$

b) $x[n] = a^n u[n]$; $h[n] = b^n u[n]$

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{K=-\infty}^{\infty} a^K u[K]\, b^{n-K} u[n-K] \\ &= \left(b^n \sum_{K=0}^{n} \left(\frac{a}{b}\right)^K\right) u[n] \\ &= b^n \left(\frac{1 - \left(\frac{a}{b}\right)^{n+1}}{1 - \left(\frac{a}{b}\right)}\right) u[n] \end{aligned}$$

$$\boxed{y[n] = \frac{b^{n+1} - a^{n+1}}{b-a}\, u[n]}$$

8. c) From 8b,

$$a^n u[n] * b^n u[n] = \frac{b^{n+1} - a^{n+1}}{b-a} u[n]$$

$$\begin{aligned}
y[n] &= x[n] * h[n] \\
&= a^n u[n] * a^n u[n] \\
&= \lim_{b \to a} a^n u[n] * b^n u[n] \\
&= \lim_{b \to a} \frac{b^{n+1} - a^{n+1}}{b-a} u[n] \\
&= \lim_{b \to a} \frac{(n+1) b^n}{1} u[n] \qquad \text{using L'Hospital}
\end{aligned}$$

$$\boxed{y[n] = (n+1) a^n u[n]}$$

d) $x[n] = \{1, \underset{\uparrow}{1}, 0, 1, 1\}$ $\quad h[n] = \{\underset{\uparrow}{3}, 2, 1, 0, -1\}$

$$\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \{3, \underset{\uparrow}{2}, 1, 0, -1\} + \{\underset{\uparrow}{3}, 2, 1, 0, -1\} \\
&\quad + \{\underset{\uparrow}{0}, 0, 3, 2, 1, 0, -1\} + \{\underset{\uparrow}{0}, 0, 0, 3, 2, 1, 0, -1\}
\end{aligned}$$

$$\boxed{y[n] = \{3, \underset{\uparrow}{5}, 3, 4, 4, 2, 1, -1, -1\}}$$

9. a) A system $T(\,)$ is linear if for some inputs x_1, x_2 with outputs y_1, y_2 respectively and a, b scalars $T(ax_1 + bx_2) = ay_1 + by_2$.

$$T(ax_1 + bx_2) = \sum_{K=-\infty}^{n} (1/2)^{n-K} (ax_1(K) + bx_2(K))$$

$$= \sum_{K=-\infty}^{n} \left((1/2)^{n-K} ax_1(K) + (1/2)^{n-K} bx_2(K)\right)$$

$$= \sum_{K=-\infty}^{n} \left(a(1/2)^{n-K} x_1(K)\right) + \sum_{K=-\infty}^{n} \left(b(1/2)^{n-K} x_2(K\right.$$

$$= a\sum_{K=-\infty}^{n} \left((1/2)^{n-K} x_1(K)\right) + b\sum_{K=-\infty}^{n} \left((1/2)^{n-K} x_2(K)\right)$$

$$= ay_1(n) + by_2(n)$$

$\therefore$ The system is linear. Q.E.D.

b) A system $T(\,)$ is time invariant if for some input x with output y, the system response to a shifted input is an equally shifted output. i.e. $T(x[n-N]) = y[n-N]$.

$$T(x[n-N]) = \sum_{K=-\infty}^{n} (1/2)^{n-K} x(K-N)$$

Let $l = K - N$, then

$$T(x[n-N]) = \sum_{l=-\infty}^{n-N} (1/2)^{n-(l+N)} x(l)$$

$$= \sum_{l=-\infty}^{n-N} (1/2)^{(n-N)-l} x(l)$$

$$T(x[n-N]) = y[n-N]$$

$\therefore$ The system is time invariant. Q.E.D.

9. c) $y[n] = \sum_{K=-\infty}^{n} (1/2)^{n-K} x[K]$

$= \sum_{K=-\infty}^{\infty} (1/2)^{n-K} u[n-K]\, x[K]$

And since the system is LTI,

$y[n] = h[n] * x[n]$

$= \sum_{K=-\infty}^{\infty} x[K]\, h[n-K]$

$\Rightarrow \boxed{h[n] = (\tfrac{1}{2})^n u[n]}$

d) The system is BIBO stable $\Leftrightarrow$

$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

$\sum_{n=-\infty}^{\infty} \left| \tfrac{1}{2}^n u[n] \right| = \sum_{n=0}^{\infty} \tfrac{1}{2}^n = \frac{1}{1-\frac{1}{2}} = 2 < \infty$

$\boxed{\therefore \text{ The system is BIBO stable.}}$

e) $H(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})}$

Since: $h[n] = (\tfrac{1}{2})^n u[n]$

$y[n] = \tfrac{1}{2} y[n-1] + x[n]$

$\Rightarrow Y(e^{j\omega}) = \tfrac{1}{2} e^{-j\omega} Y(e^{j\omega}) + X(e^{j\omega})$

$Y(e^{j\omega})(1 - \tfrac{1}{2} e^{-j\omega}) = X(e^{j\omega})$

$$\boxed{H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-\frac{1}{2}e^{-j\omega}}}$$

9. f) $y[n] = ?$; $x[n] = \cos(\pi n)$

From problem 4c, when $x[n] \cos(w_0 n)$

$$y[n] = \frac{1}{4\pi}\left(H(e^{jw_0}) e^{jw_0 n} + H(e^{-jw_0}) e^{-jw_0 n} \right)$$

In this case $w_0 = \pi$, $H(e^{j\pi}) = H(e^{-j\pi})$.

$$H(e^{j\pi}) = \frac{1}{1-\frac{1}{2}e^{-j\pi}} = \frac{1}{1-(\frac{1}{2})(-1)} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$

$$y[n] = \frac{1}{4\pi}\left(\frac{2}{3} e^{j\pi n} + \frac{2}{3} e^{-j\pi n} \right)$$

$$y[n] = \left(\frac{1}{2\pi}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(e^{j\pi n} + e^{-j\pi n}\right)$$

$$\boxed{y[n] = \frac{1}{3\pi}\cos(\pi n)}$$

g) $$y[n] = \sum_{K=-\infty}^{n} (1/2)^{n-K} x[K]$$

$$= \sum_{K=-\infty}^{n-1} (1/2)^{n-K} x[K] + x[n]$$

$$= \frac{1}{2}\sum_{K=-\infty}^{n-1} (1/2)^{(n-1)-K} x[K] + x[n]$$

$$\boxed{y[n] = \frac{1}{2} y[n-1] + x[n]}$$