ECE 438 HW#2 Solutions Fall 2007

1. $y[n] = x[n] + 2x[n-1] + 0.5y[n-1]$

a. $h[n] = 0.5^n u[n] + 2(0.5)^{n-1} u[n-1]$

let $x[n] = \delta[n]$, then $y[n] = h[n]$

$h[-1] = 0$

$h[0] = \delta[n] + 2(0) + 0.5(0) = 1$

$h[1] = 0.5(1) + 2(1)$

$h[2] = 0.5(0.5(1) + 2) = 0.5^2 + 0.5(2)$

$h[3] = 0.5(0.5^2 + 0.5(2)) = 0.5^3 + 0.5^2(2)$

$\vdots$

$h[n] = 0.5^n u[n] + 2(0.5)^{n-1} u[n-1]$

b. $y[n] = ?$ when $x[n] = u[n]$

$$y[n] = h[n] * x[n]$$
$$= (0.5^n u[n] + 2(0.5)^{n-1} u[n-1]) * u[n]$$

① $$0.5^n u[n] * u[n] = \sum_{k=-\infty}^{\infty} 0.5^k u[k]\, u[n-k]$$
$$= \left(\sum_{k=0}^{n} 0.5^k\right) u[n]$$
$$= \left(\frac{1-0.5^{n+1}}{1-0.5}\right) u[n] = 2(1-(0.5)^{n+1}) u[n]$$

② $$2(0.5)^{n-1} u[n-1] * u[n] = 2(0.5)^n u[n] * \delta[n-1] * u[n]$$
$$= 2(0.5)^n u[n] * u[n] * \delta[n-1]$$
$$= 2(2(1-(0.5)^{n+1}) u[n]) * \delta[n-1]$$
$$= 4(1-(0.5)^n) u[n-1]$$

$$\boxed{y[n] = \underbrace{2(1-(0.5)^{n+1}) u[n]}_{\text{From ①}} + \underbrace{4(1-(0.5)^n) u[n-1]}_{\text{From ②}}}$$

1. c. $x[n] = 0.25^n u[n]$ $\qquad y[n] = ?$

$$y[n] = h[n] * x[n]$$
$$= (0.5^n u[n] + 2(0.5)^{n-1} u[n-1]) * ((0.25)^n u[n])$$

① $(0.5^n u[n] * 0.25^n u[n]) = 0.25^n \left(\dfrac{1 - \left(\frac{0.5}{0.25}\right)^{n+1}}{1 - \left(\frac{0.5}{0.25}\right)} \right) u[n]$

$$= 0.25^n \left(\frac{1-(2)^{n+1}}{-1} \right) u[n]$$

$$= 0.25^n ((2)^{n+1} - 1) u[n]$$

② $(2(0.5)^{n-1} u[n-1] * 0.25^n u[n])$

$$= 2\delta[n-1] * (0.5^n u[n] * 0.25^n u[n])$$
$$= 2\delta[n-1] * \underbrace{0.25^n ((2)^{n+1} - 1) u[n]}_{\text{From ①}}$$
$$= 2(0.25^{n-1}(2^n - 1)) u[n-1]$$

$$\boxed{y[n] = 0.25^n (2^{n+1} - 1) u[n] + 2(0.25^{n-1}(2^n - 1)) u[n-1]}$$

2\. a) $e^{-t}u(t)$

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{0}^{\infty} e^{-t} e^{-j2\pi f t}\, dt = \int_{0}^{\infty} e^{-(1+j2\pi f)t}\, dt$$

$$= \frac{-1}{1+j2\pi f} e^{-(1+j2\pi f)t} \Big|_{0}^{\infty}$$

$$= \frac{-1}{1+j2\pi f}\left(0 - 1\right) = \boxed{\frac{1}{1+j2\pi f}}$$

b) $x(t) = e^{j\omega_0 t}$

$$= 1 \cdot e^{j\omega_0 t}$$

$$= x_1(t) \cdot e^{j\omega_0 t}$$

$$x_1(t)e^{j\omega_0 t} \overset{CTFT}{\Longleftrightarrow} X_1\left(f - \frac{\omega_0}{2\pi}\right)$$

$$x_1(t) = 1 \overset{CTFT}{\Longleftrightarrow} \delta(f)$$

$$X(f) = X_1\left(f - \frac{\omega_0}{2\pi}\right)$$

$$\boxed{X(f) = \delta\left(f - \frac{\omega_0}{2\pi}\right)}$$

c) $x(t) = \text{rect}(t)e^{j6\pi t}$

$$= x_1(t)\, e^{j6\pi t}$$

$$X(f) = X_1(f - 3)$$

$$\text{rect}(t) \overset{CTFT}{\Longleftrightarrow} \text{sinc}(f)$$

$$\boxed{X(f) = \text{sinc}(f - 3)}$$

2. d) $x(t) = \text{sinc}(t)\cos(2\pi f_0 t)$

$$= \text{sinc}(t)\frac{1}{2}\left(e^{-j2\pi f_0 t} + e^{j2\pi f_0 t}\right)$$

$$= \frac{1}{2}x_1(t)\left(e^{-j2\pi f_0 t} + e^{j2\pi f_0 t}\right)$$

$$X(f) = \frac{1}{2}\left(X_1(f+f_0) + X_1(f-f_0)\right)$$

$$x_1(t) = \text{sinc}(t) \overset{CTFT}{\Longleftrightarrow} \text{rect}(f)$$

$$\boxed{X(f) = \frac{1}{2}\left(\text{rect}(f+f_0) + \text{rect}(f-f_0)\right)}$$

e) $x(t) = \cos(2\pi f_0 t)\,\text{rect}(t)$

$$= \frac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)x_1(t)$$

$$X(f) = \frac{1}{2}\left(X_1(f-f_0) + X_1(f+f_0)\right)$$

$$\text{rect}(t) \overset{CTFT}{\Longleftrightarrow} \text{sinc}(f)$$

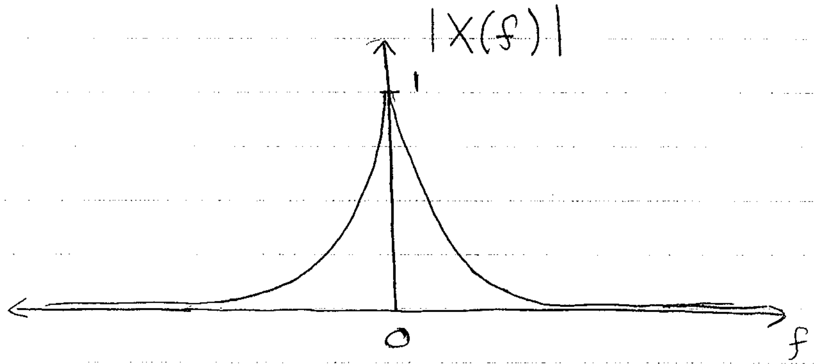
$$\boxed{X(f) = \frac{1}{2}\left(\text{sinc}(f-f_0) + \text{sinc}(f+f_0)\right)}$$

2. a. $X(f) = \dfrac{1}{1+j2\pi f}$

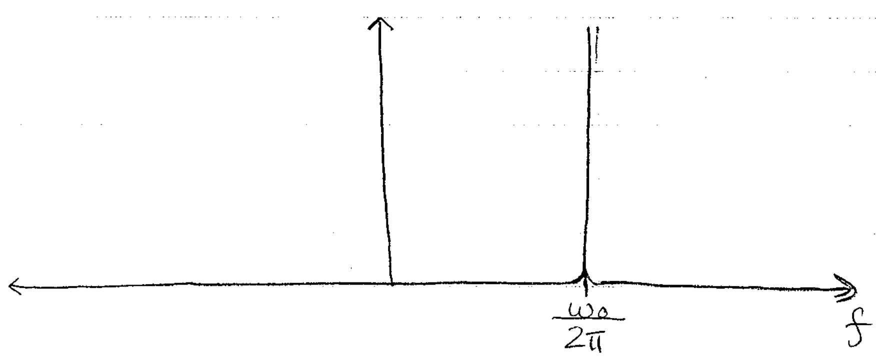
$$|X(f)| = \sqrt{X(f)X^*(f)}$$

$$= \sqrt{\left(\frac{1}{1+j2\pi f}\right)\left(\frac{1}{1-j2\pi f}\right)}$$

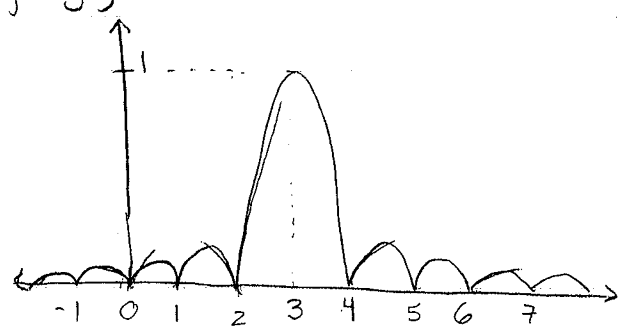
$$= \sqrt{\frac{1}{1+4\pi^2 f^2}}$$



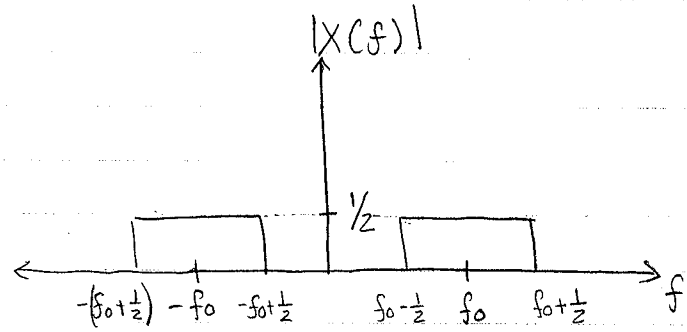
b. $X(f) = \delta\left(f - \frac{\omega_0}{2\pi}\right)$



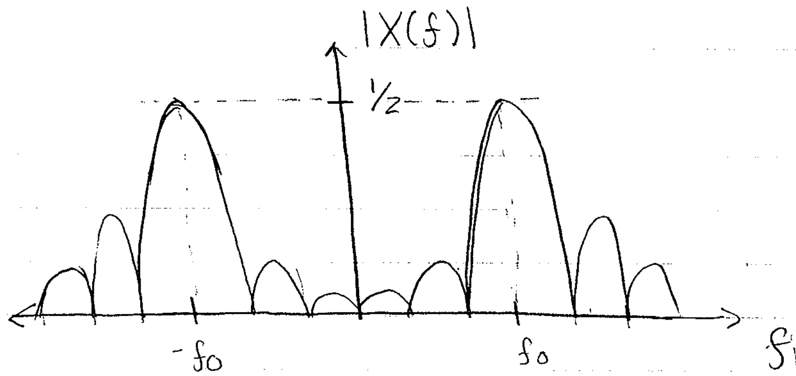
c. $X(f) = \text{sinc}\,(f-3)$



2. d.



e.



3. a. $u[n+N] - u[n-N-1] = u[n] - u[n-(N+1)]$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-N}^{N} e^{-j\omega n} = \frac{e^{j\omega N} - e^{-j\omega(N+1)}}{1-e^{-j\omega}}$$

$$= \frac{e^{-j\omega/2}\left(e^{j\omega(2N+1)/2} - e^{-j\omega(2N+1)/2}\right)}{1-e^{-j\omega}}$$

$$= \frac{j2\sin\left(\frac{2N+1}{2}(\omega)\right)}{e^{j\omega/2} - e^{-j\omega/2}}$$

$$\boxed{X(\omega) = \frac{\sin\left(\frac{2N+1}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}}$$

b. $2^n u[-n]$

$$X(\omega) = \sum_{n=-\infty}^{\infty} 2^n u[-n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{0} (2e^{-j\omega})^n$$

Let $m = -n$, then

$$X(\omega) = \sum_{m=0}^{\infty} (2e^{-j\omega})^{-m}$$

$$= \sum_{m=0}^{\infty} \left(\tfrac{1}{2}e^{j\omega}\right)^m = \boxed{\frac{1}{1-\frac{1}{2}e^{j\omega}}}$$

$$X(\omega) = \frac{1}{1-\frac{1}{2}e^{j\omega}} \cdot \frac{1-\frac{1}{2}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}} = \frac{1-\frac{1}{2}e^{-j\omega}}{1-\frac{1}{2}(e^{j\omega}+e^{-j\omega})+}$$

$$= \frac{1-\frac{1}{2}e^{-j\omega}}{\frac{5}{4}-\cos(\omega)}$$

3. c. $a^n \sin(\omega_0 n) u(n) \quad |a| < 1, \quad |\omega_0| < \pi$

$= a^n u[n] \sin[\omega_0 n]$

Let $x_1[n] = a^n u[n]$, then

$$X_1(\omega) = \frac{1}{1 - ae^{-j\omega}}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} a^n u[n] \sin[\omega_0 n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n \frac{(e^{j\omega_0 n} - e^{-j\omega_0 n})}{j2} e^{-j\omega n}$$

$$= \frac{1}{j2} \sum_{n=0}^{\infty} \left[\left(a e^{-j(\omega-\omega_0)} \right)^n - \left(a e^{-j(\omega+\omega_0)} \right)^n \right]$$

$$\boxed{X(\omega) = \frac{1}{j2} \left(\frac{1}{1 - ae^{-j(\omega-\omega_0)}} - \frac{1}{1 - ae^{-j(\omega+\omega_0)}} \right)}$$

d. $\cos(18\pi n / 7)$

$$x[n] = \cos\left(\frac{18\pi n}{7}\right) = \cos\left(\frac{4\pi n}{7} + 2\pi n\right) = \cos\left(\frac{4\pi n}{7}\right)$$

$$x[n] = \frac{1}{2}\left(e^{j\frac{4\pi n}{7}} + e^{-j\frac{4\pi n}{7}}\right)$$

$$X(\omega) = \frac{1}{2}\left[\text{DTFT}\{e^{j\frac{4\pi n}{7}}\} + \text{DTFT}\{e^{-j\frac{4\pi n}{7}}\}\right]$$

$$\boxed{X(\omega) = \frac{1}{2}\left(\delta(\omega - \tfrac{4\pi}{7}) + \delta(\omega + \tfrac{4\pi}{7})\right)}$$

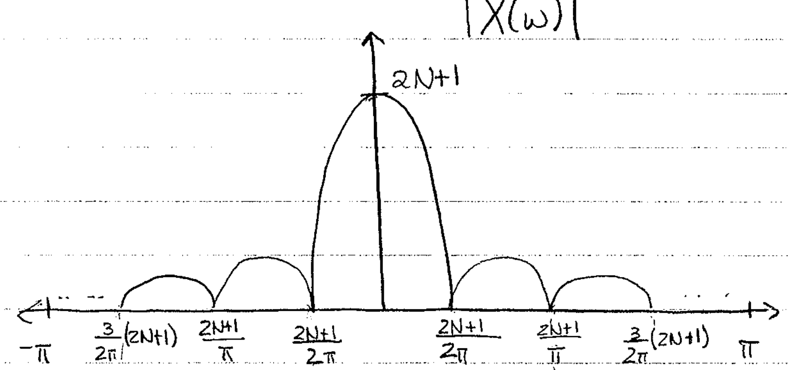
3. e. $\dfrac{\sin(\pi n/8)}{\pi n}$

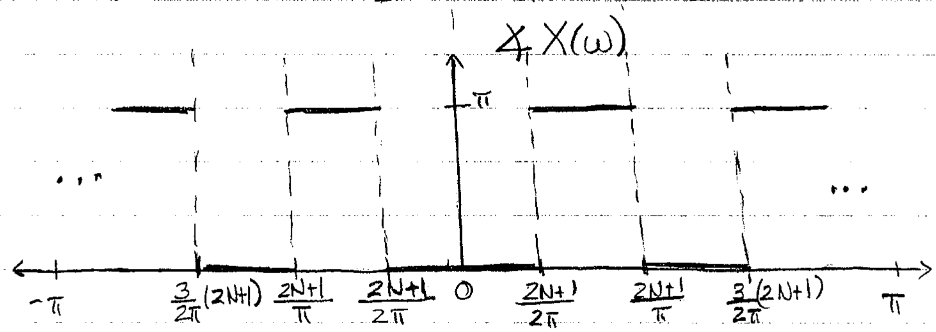
$$x[n] = \frac{\sin \omega_c n}{\pi n} \xleftrightarrow{\text{DTFT}} X(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c \le |\omega| \le \pi \end{cases}$$

$$\boxed{X(\omega) = \begin{cases} 1 & |\omega| < \frac{\pi}{8} \\ 0 & \frac{\pi}{8} \le |\omega| \le \pi \end{cases}}$$

Magnitude and Phase Plots start on next page

3. a. $X(\omega) = \dfrac{\sin\left((2N+1)\frac{\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$

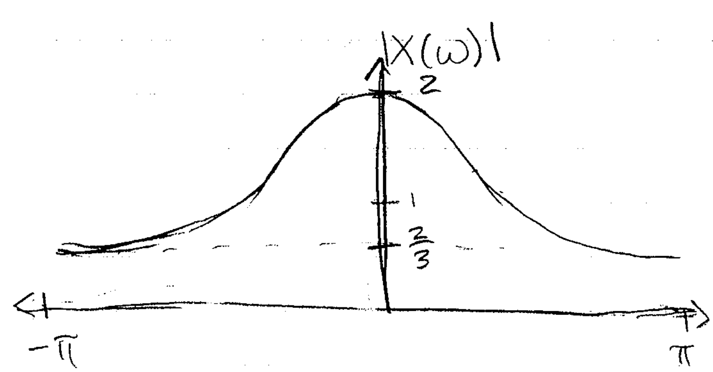




b. $X(\omega) = \dfrac{1}{1-\frac{1}{2}e^{j\omega}}$

$$|X(\omega)| = \sqrt{\left(\frac{1}{1-\frac{1}{2}e^{j\omega}}\right)\left(\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right)} = \sqrt{X(\omega)X^*(\omega)}$$

$$= \sqrt{\frac{1}{1-\frac{1}{2}(e^{j\omega}+e^{-j\omega})+\frac{1}{4}}} = \sqrt{\frac{1}{\frac{5}{4}-\cos(\omega)}}$$

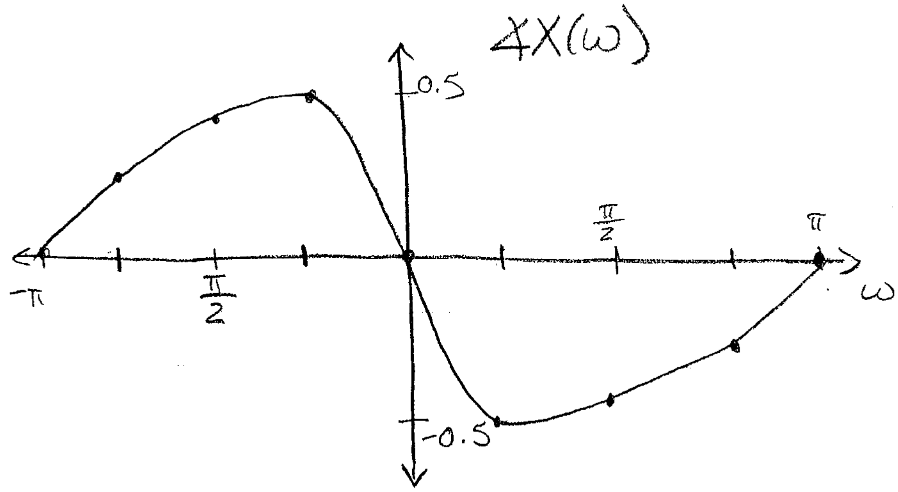


3. b. $X(\omega) = \frac{1}{1-\frac{1}{2}e^{j\omega}} \cdot \frac{1-\frac{1}{2}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}} = \frac{1-\frac{1}{2}e^{-j\omega}}{\frac{5}{4}-\cos\omega}$

$$\measuredangle X(\omega) = \measuredangle\left(\tfrac{5}{4}-\cos\omega\right) + \measuredangle\left(1-\tfrac{1}{2}e^{-j\omega}\right)$$
$$= 0 + \measuredangle\left(1-\tfrac{1}{2}e^{-j\omega}\right)$$

$$1-\tfrac{1}{2}e^{-j\omega} = 1-\tfrac{1}{2}\cos\omega - j\tfrac{1}{2}\sin\omega$$
$$\measuredangle X(\omega) = \tan^{-1}\left(\frac{-\frac{1}{2}\sin\omega}{1-\frac{1}{2}\cos\omega}\right)$$
$$= \tan^{-1}\left(\frac{\sin\omega}{\cos\omega-2}\right)$$

ω	$\sin\omega$	$\cos\omega$	$\cos\omega-2$	$\measuredangle X(\omega)$
0	0	1	-1	0 $(-\pi)$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}-2<0$	-0.5
$\frac{\pi}{2}$	1	0	-2	-0.4656
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}-2$	-0.2555
π	0	-1	-3	0
$-\frac{\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}-2$	0.5
$-\frac{\pi}{2}$	-1	0	-2	0.4656
$-\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}-2$	0.2555
$-\pi$	0	-1	-3	0



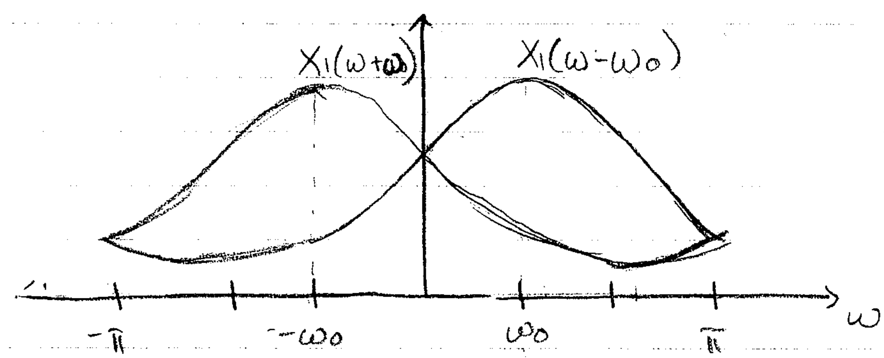
3.c. $$X(\omega) = \frac{1}{j2}\left(\frac{1}{1-ae^{-j(\omega-\omega_0)}} - \frac{1}{1-ae^{-j(\omega+\omega_0)}}\right)$$

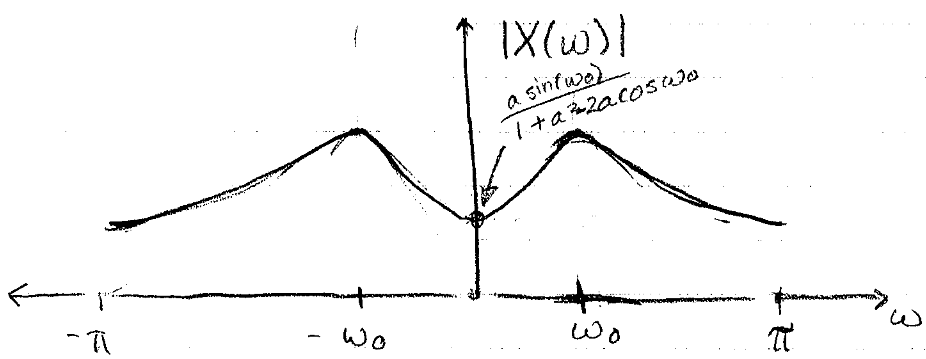
$$X(\omega) = \frac{1}{j2}\left(\frac{1-ae^{+j(\omega-\omega_0)}}{1+a^2-2a\cos(\omega-\omega_0)} - \frac{1-ae^{j(\omega+\omega_0)}}{1+a^2+2a\cos(\omega+\omega_0)}\right)$$

$$= \frac{1}{j2}\left(X_1(\omega+\omega_0) - X_1(\omega+\omega_0)\right)$$

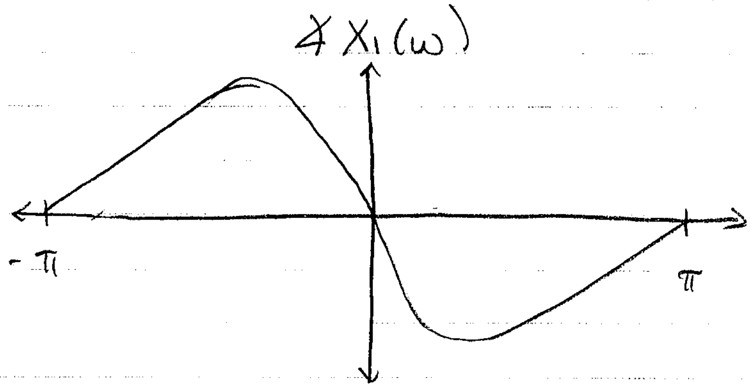
$|X(\omega)|$

ω	$X_1(\omega-\omega_0)$
ω_0	$(1-a)/(1-a)^2$
$\omega_0+\frac{\pi}{2}$	$(1-ja)/(1+a^2)$
$\omega_0+\pi$	$(1+a)/(1+a)^2$
$\omega_0-\frac{\pi}{2}$	$(1+ja)/(1+a^2)$
$\omega_0-\pi$	$(1+a)/(1+a)^2$

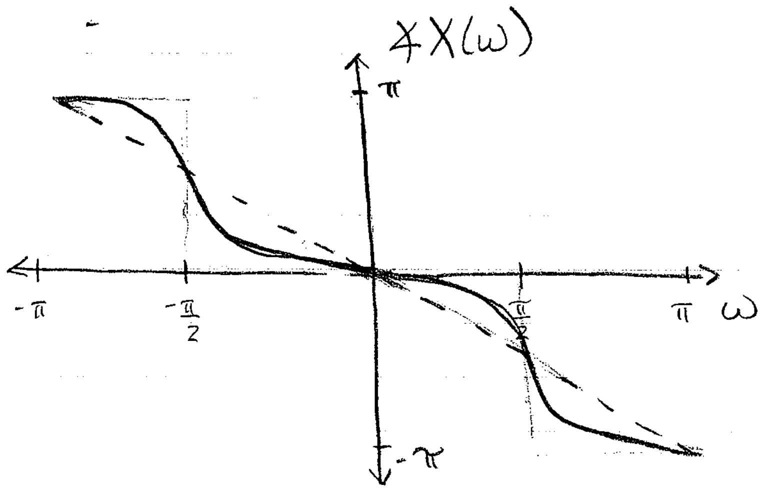




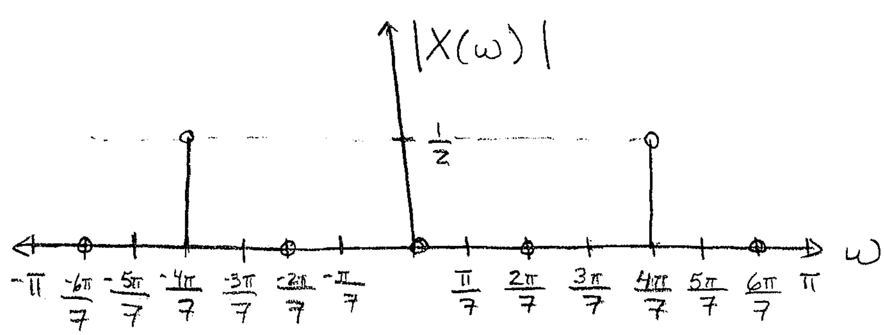
3. c.

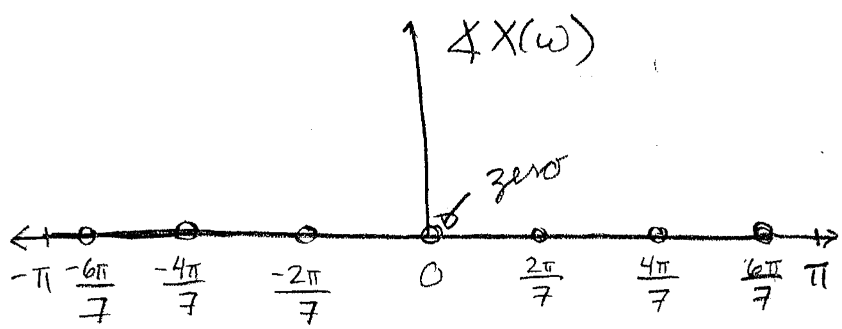


$\measuredangle \frac{1}{j2}\left(X_1(\omega-\omega_0) - X_1(\omega+\omega_0)\right)$

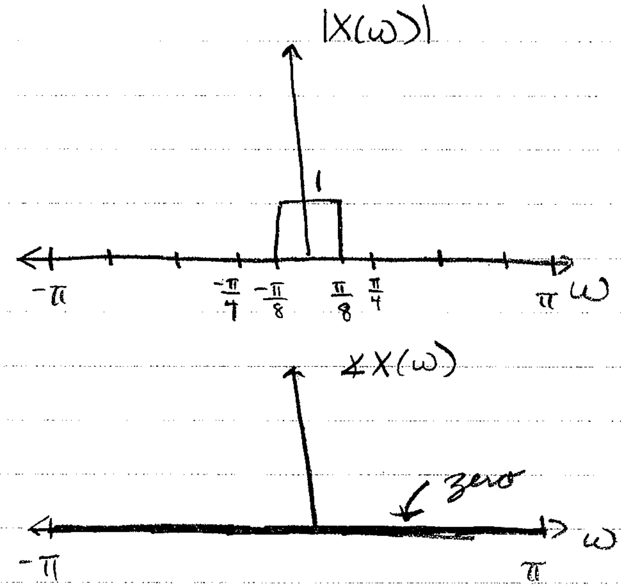


d.





3. e.



4. a. $x(n-N)\,e^{j\omega_0 n}$

$$\sum_{n=-\infty}^{\infty} x(n-N)\, e^{j\omega_0 n}\, e^{-j\omega n} = ?$$

Let $l = n-N$, then

$$\sum_{n=-\infty}^{\infty} x(n-N)e^{j\omega_0 n}e^{-j\omega n} = \sum_{l=-\infty}^{\infty} x(l)\, e^{-j(\omega-\omega_0)(l+N)}$$

$$= e^{-j(\omega-\omega_0)N} \sum_{l=-\infty}^{\infty} x(l)\, e^{-j(\omega-\omega_0)l}$$

$$\boxed{\text{DTFT}\{x(n-N)e^{j\omega_0 n}\} = e^{-j(\omega-\omega_0)N}\, X(\omega-\omega_0)}$$

b. DTFT$\{x^*(-n)\}$

$x(n) = x_R(n) + j x_I(n)$
$X(\omega) = X_R(\omega) + j X_I(\omega)$
$e^{-j\omega n} = \cos\omega n - j\sin\omega n$

$$\text{DTFT}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} (x_R(n) + j x_I(n))(\cos\omega n - j\sin\omega n)$$

$$= \sum_{n=-\infty}^{\infty} \Big(x_R(n)\cos(\omega n) + x_I(n)\sin\omega n$$
$$+ j\big(x_I(n)\cos\omega n - x_R(n)\sin\omega n\big)\Big)$$

$$X_R(\omega) = \sum_{n=-\infty}^{\infty} x_R(n)\cos\omega n + x_I \sin\omega n$$

$$X_I(\omega) = \sum_{n=-\infty}^{\infty} x_I(n)\cos\omega n - x_R(n)\sin\omega n$$

4. b. $\text{DTFT}\{x^*(-n)\} = \;?$

$$\text{DTFT}\{x^*(-n)\} = \sum_{n=-\infty}^{\infty} x^*(-n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(x_R(-n) - j\, x_I(-n)\right)\left(\cos\omega n - j\sin\omega n\right)$$

$$= \sum_{n=-\infty}^{\infty} \Big(x_R(-n)\cos(\omega n) - x_I(-n)\sin(\omega n) - j\left(x_R(-n)\sin\omega n + x_I(-n)\cos\omega n\right)\Big)$$

Let $l = -n$, then

$$\text{DTFT}\{x^*(-n)\} = \sum_{l=-\infty}^{\infty} \Big(x_R(l)\cos(\omega l) + x_I(l)\sin(\omega l) + j\left(x_R(l)\sin\omega l - x_I(l)\cos\omega l\right)\Big)$$

$$= X_R(\omega) - j X_I(\omega)$$

$$\boxed{\text{DTFT}\{x^*(-n)\} = X^*(\omega)}$$

4. c. $x(n)\,y(n) \overset{DTFT}{\Longleftrightarrow}$?

$$y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(\omega)e^{j\omega n}\,d\omega \quad = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(\mu)e^{j\mu n}\,d\mu$$

$$DTFT\{x(n)y(n)\} = \sum_{n=-\infty}^{\infty} x(n)y(n)e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} Y(\mu)e^{j\mu n}\,d\mu\right)e^{-j\omega n}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}\,Y(\mu)\,e^{j\mu n}\,d\mu$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega-\mu)n}\,Y(\mu)\,d\mu$$

$$\boxed{DTFT\{x(n)y(n)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega-\mu)\,Y(\mu)\,d\mu}$$

d. $DTFT\{x(n)^2\} = ?$ From 4c.

$$\boxed{DTFT\{x(n)\,x(n)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega-\mu)X(\mu)\,d\mu}$$

5. $y[n] = \frac{1}{4}\{x[n] - 2x[n-1] + x[n-2]\}$

a. $H(\omega) = ?$

$$h[n] = \frac{1}{4}\{\delta[n] - 2\delta[n-1] + \delta[n-2]\}$$

$$\boxed{H(\omega) = \frac{1}{4}(1 - 2e^{-j\omega} + e^{-j2\omega})}$$

b. $$H(\omega) = \frac{1}{4}e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2}) - \frac{1}{4}(e^{-j\omega} - e^{-j2\omega})$$

$$= \frac{1}{4}e^{-j\omega/2}(j2)\sin(\omega/2) - \frac{1}{4}e^{-j3\omega/2}(e^{j\omega/2} - e^{-j\omega/2})$$

$$= \frac{1}{4}[e^{-j\omega/2}(j2)\sin(\omega/2) - e^{-j3\omega/2}(j2)\sin(\omega/2)]$$

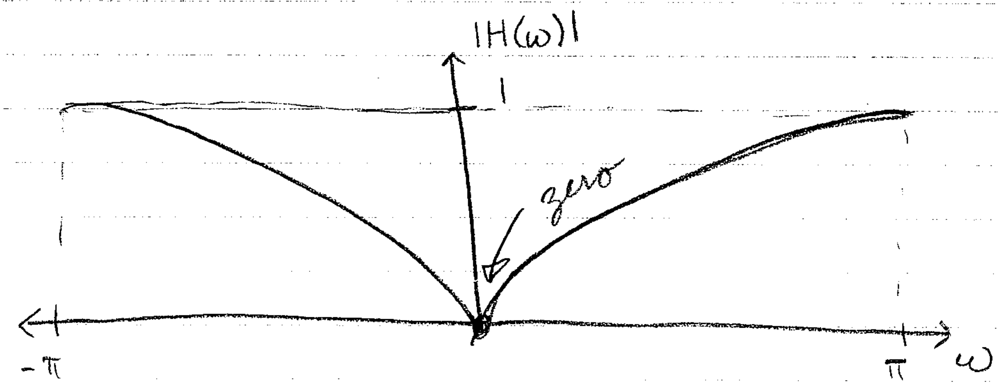
$$= \frac{1}{4}(j2)\sin(\omega/2)[e^{-j\omega/2} - e^{-j3\omega/2}]$$

$$= \frac{1}{4}(j2)e^{-j\omega}\sin(\omega/2)[e^{j\omega/2} - e^{-j\omega/2}]$$

$$= \frac{1}{4}(j2)^2 e^{-j\omega}\sin(\omega/2)^2$$

$$= e^{-j(\omega+\pi)}\sin(\omega/2)^2$$

$$\boxed{|H(\omega)| = \sin(\omega/2)^2}$$



5. c. $\arg\{H(\omega)\} = -(\omega + \pi)$

d.