The Continuous Time Fourier Transform (CTFT)

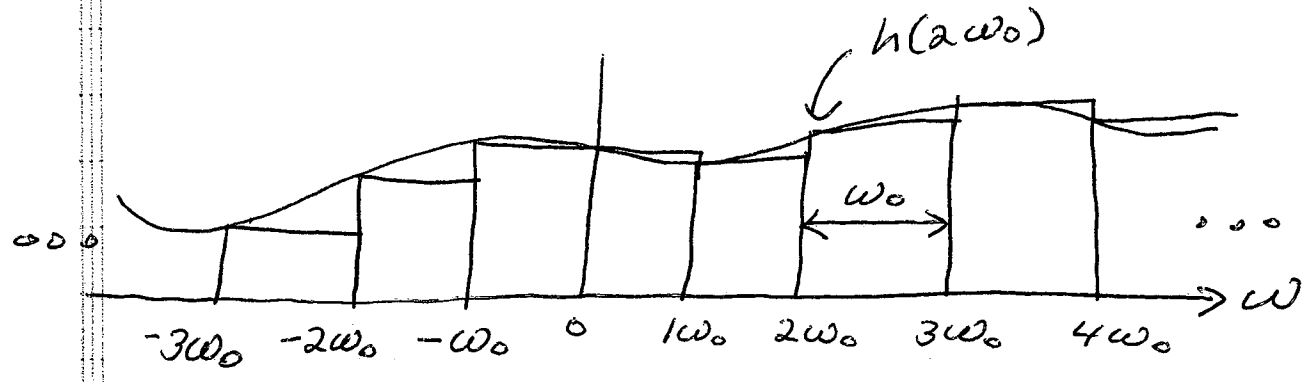
- Object
  - Extend CTFS to functions that are <u>not</u> periodic.
  - Assume $x(t)$ decays to 0 as $|t|$ becomes large
  - Take limit a period $T \to \infty$

- For now on we will use $\omega_0 = \frac{2\pi}{T}$ as period of CTFS.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j\omega_0 k t} dt$$

Definition of integral



$$\int_{-\infty}^{\infty} h(\omega)\, d\omega \triangleq \lim_{\omega_0 \to 0} \sum_{k=-\infty}^{\infty} \omega_0 h(k\omega_0)$$

So we have

$$x(t) = \lim_{\omega_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \omega_0 X_T(\omega_0 k) e^{j\omega_0 k t}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\lim_{\omega_0 \to 0} X_T(\omega) \right) e^{j\omega t}\, d\omega$$

<u>But</u> $\lim_{\omega_0 \to 0} X_T(\omega) = \lim_{\omega_0 \to 0} X_{\frac{2\pi}{\omega_0}}(\omega)$

$$= \lim_{\omega_0 \to 0} \int_{-\pi/\omega_0}^{\pi/\omega_0} x(t) e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

Define

$$X_T(\omega) = \int_{-T/2}^{T/2} x(t) e^{-j\omega t} dt$$

Then

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-j\omega_0 k t} dt$$

$$= \frac{1}{T} X_T(\omega_0 k)$$

By the Fourier series expansion, we know

For $|t| \le T/2$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T} X_T(\omega_0 k) e^{j\omega_0 k t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \omega_0 X_T(\omega_0 k) e^{j\omega_0 k t}$$

So we have

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\lim_{\omega_0 \to 0} X_T(\omega)\right)e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\,d\omega$$

where

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\,dt$$

This yields the CTFT

CTFT

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\,dt$$

Inverse CTFT

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\,d\omega$$

Example:

1) $x(t) = e^{-at}u(t) \qquad a>0$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}\,dt$$

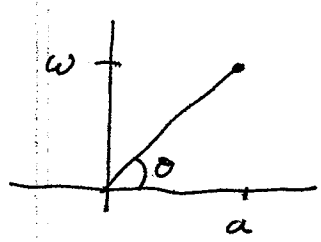
$$= \int_{-\infty}^{\infty} e^{-(j\omega+a)t}u(t)\,dt$$

$$= \int_{0}^{\infty} e^{-(j\omega+a)t}\,dt$$

$$= 0 - \left(\frac{1}{-(j\omega+a)}\right)$$

$$= \frac{1}{j\omega+a}$$

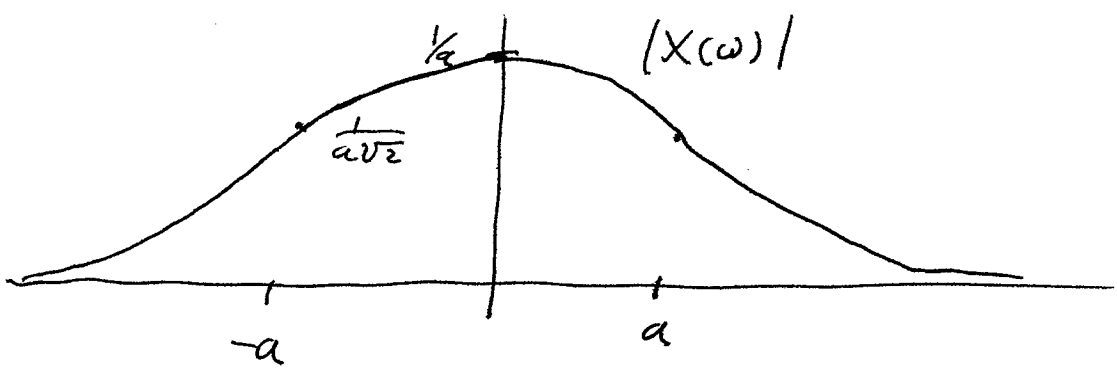
$$\boxed{e^{-at}u(t) \overset{CTFT}{\Longleftrightarrow} \frac{1}{j\omega+a}}$$



$$|X(\omega)| = \sqrt{\frac{1}{\omega^2+a^2}}$$

$$\angle X(\omega) = -\left(\tan^{-1}\left(\frac{\omega}{a}\right)\right)$$

$$X(\omega) = \frac{1}{\sqrt{\omega^2+a^2}}\, e^{-j\tan^{-1}\left(\frac{\omega}{a}\right)}$$



Example 2

$$x(t) = \delta(t)$$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= 1$$

$$\boxed{\delta(t) \overset{CTFT}{\Longleftrightarrow} 1}$$

Example

$$x(t) = \text{rect}(t)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-1/2}^{1/2} e^{-j\omega t}\, dt$$

$$= \frac{1}{-j\omega} e^{-j\omega t} \Big|_{-1/2}^{1/2}$$

$$= \frac{1}{-j\omega} \left(e^{-j\omega/2} - e^{+j\omega/2} \right)$$

$$= \frac{2}{\omega} \cdot \frac{e^{j\omega/2} - e^{-j\omega/2}}{2j}$$

$$= \frac{1}{\omega/2} \sin(\omega/2)$$

$$= \text{sinc}(\omega/2\pi)$$

$$\boxed{\text{rect}(t) \overset{CTFT}{\Longleftrightarrow} \text{sinc}(\omega/2\pi)}$$

Fourier Transforms and LTI systems

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \underbrace{\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega t}\, dt}\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau} \underbrace{\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega(t-\tau)}\, dt}_{H(\omega)}\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, H(\omega)\, d\tau$$

$$= X(\omega)\, H(\omega)$$

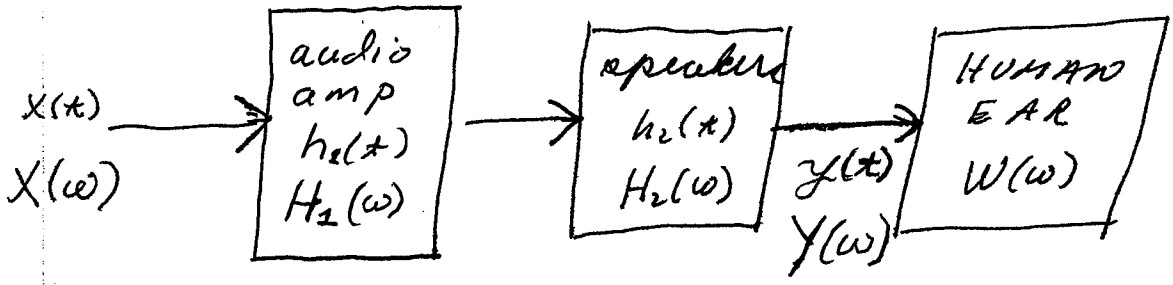
The Fourier transform turns convolution into multiplication

$$X(\omega) \longrightarrow \boxed{H(\omega)} \longrightarrow Y(\omega) = X(\omega)H(\omega)$$

$H(\omega)$ is called the systems:

- Frequency response
- Transfer function

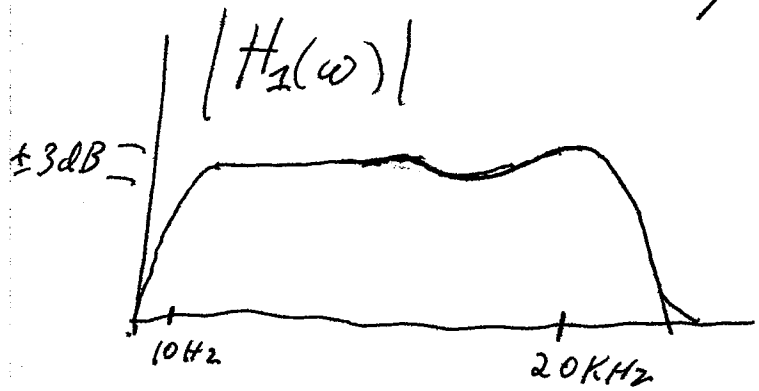
Example: Audio System

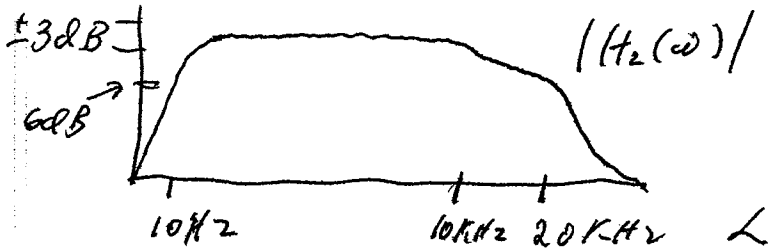


$$y(t) = x(t) * h_1(t) * h_2(t)$$

$$Y(\omega) = X(\omega) * H_1(\omega) * H_2(\omega)$$

frequency response of amp





Total system is

$$H_1(\omega)\, H_2(\omega) = H(\omega)$$

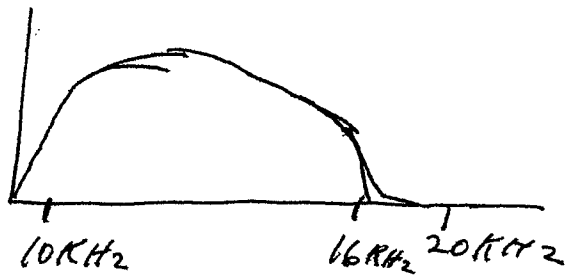
$$|H(\omega)| = |H_1(\omega)|\,|H_2(\omega)|$$

$$\pm\ 3 + 6 = 9\,dB$$

What about phase?

$$\angle H(\omega) = (\angle H_1(\omega)) + (\angle H_2(\omega))$$

$W(\omega)$



Properties of Fourier Transform

P1 - Linearity

$$x_1(t) \overset{CTFT}{\Longleftrightarrow} X_1(\omega)$$

$$x_2(t) \overset{CTFT}{\Longleftrightarrow} X_2(\omega)$$

then

$$\alpha x_1(t) + \beta x_2(t) \overset{CTFT}{\Longleftrightarrow} \alpha X_1(\omega) + \beta X_2(\omega)$$

P2 - Time Reversal

$$x(t) \overset{CTFT}{\Longleftrightarrow} X(\omega)$$

then

$$x(-t) \overset{CTFT}{\Longleftrightarrow} X(-\omega)$$

P3 - Time shift

$$x(t) \overset{CTFT}{\Longleftrightarrow} X(\omega)$$

$$x(t-t_0) \overset{CTFT}{\Longleftrightarrow} X(\omega)\, e^{-j\omega t_0}$$

P4 - Time Scaling

$$x(at) \overset{CTFT}{\Longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right).$$

P5 – Conjugate Symmetry

Assume $x(t)$ is real with

$$x(t) \overset{CTFT}{\Longleftrightarrow} X(\omega)$$

then

$$X(\omega) = X^*(-\omega)$$

P6 – Duality

Assume that

$$g(t) \overset{CTFT}{\Longleftrightarrow} f(\omega)$$

then

$$f(t) \overset{CTFT}{\Longleftrightarrow} 2\pi g(-\omega)$$

P7 - Frequency Shift

$$x(t) \overset{CTFT}{\Longleftrightarrow} X(\omega)$$

$$x(t)e^{j\omega_0 t} \overset{CTFT}{\Longleftrightarrow} X(\omega-\omega_0)$$

P8 - Time domain multiplication

$$x(t) \overset{CTFT}{\Longleftrightarrow} X(\omega)$$

$$y(t) \overset{CTFT}{\Longleftrightarrow} Y(\omega)$$

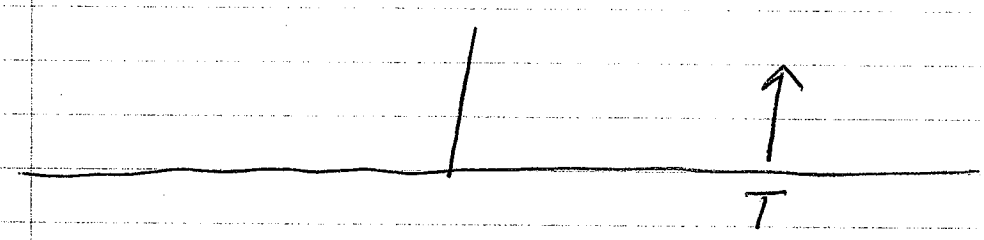
$$x(t)\cdot y(t) \overset{CTFT}{\Longleftrightarrow} \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\,Y(\omega-\lambda)\,d\lambda$$

$$= \frac{1}{2\pi} X(\omega) * Y(\omega)$$

$$\begin{pmatrix}\text{Time domain}\\ \text{multiplication}\end{pmatrix} \Longleftrightarrow \begin{pmatrix}\text{Freq. Domain}\\ \text{convolution}\end{pmatrix}$$

Example

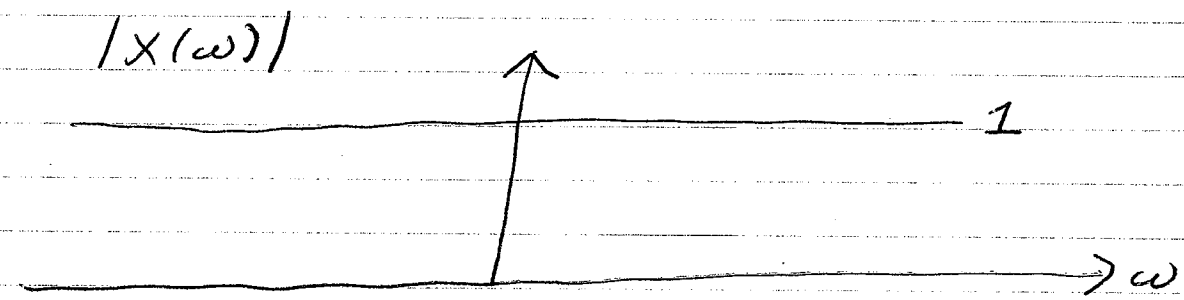
$$x(t) = \delta(t-T)$$

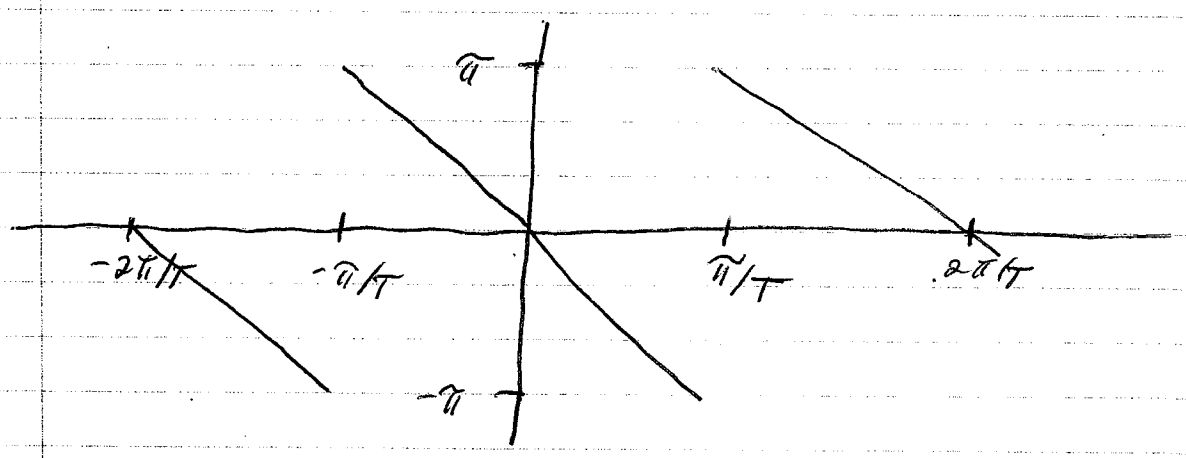


$$\delta(t) \overset{CTFT}{\Longleftrightarrow} 1$$

$$\delta(t-T) \overset{CTFT}{\Longleftrightarrow} e^{-j\omega T}$$

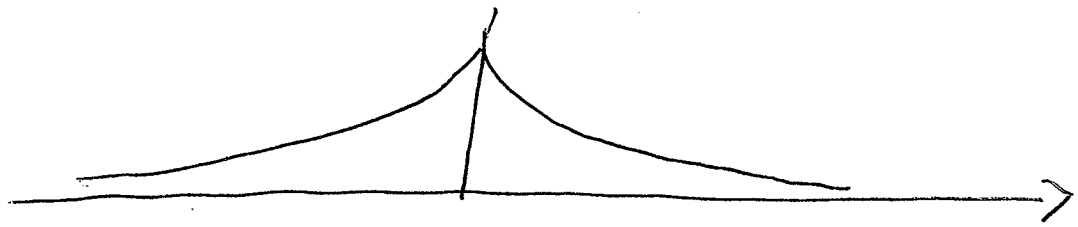
$$X(\omega) = e^{-j\omega T}$$





Example

$$x(t) = e^{-a|t|}$$



$$x(t) = e^{-at}u(t) + e^{at}u(-t)$$

$$e^{-at}u(t) \overset{CTFT}{\Longleftrightarrow} \frac{1}{j\omega + a}$$

$$e^{at}u(-t) \overset{CTFT}{\Longleftrightarrow} \frac{1}{-j\omega + a}$$

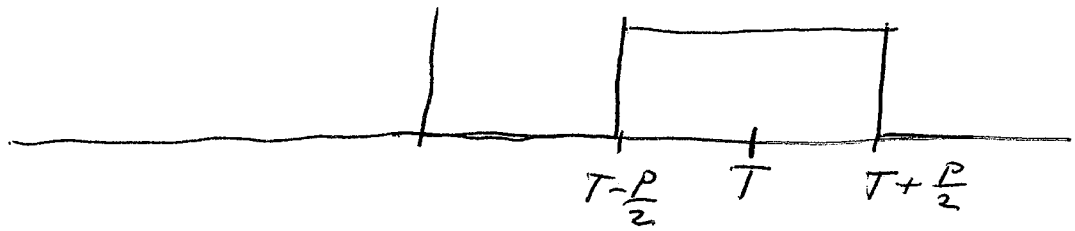
$$e^{-a|t|} \overset{CTFT}{\Longleftrightarrow} \frac{1}{j\omega + a} + \frac{1}{-j\omega + a}$$

$$= \frac{-j\omega + a + j\omega + a}{(j\omega + a)(-j\omega + a)} = \frac{2a}{\omega^2 + a^2}$$

$$e^{-a|t|} \overset{CTFT}{\Longleftrightarrow} \frac{2a}{\omega^2 + a^2}$$

Example

$$x(t) = \text{rect}\left(\frac{t-T}{p}\right)$$



$$\text{rect}(t) \overset{CTFT}{\Longleftrightarrow} \text{sinc}(\omega/2\pi)$$

$$\text{rect}(t/p) \overset{CTFT}{\Longleftrightarrow} p\,\text{sinc}(p\omega/2\pi)$$

$$\text{rect}\left(\frac{t-T}{p}\right) \overset{CTFT}{\Longleftrightarrow} p\,\text{sinc}(p\omega/2\pi)\,e^{-j\omega T}$$

Example

$$x(t) = \text{sinc}(t)$$

We know that

$$\text{rect}(t) \overset{CTFT}{\Longleftrightarrow} \text{sinc}(\omega/2\pi)$$

So therefore we know that

$$\text{sinc}(t/2\pi) \overset{CTFT}{\Longleftrightarrow} 2\pi\,\text{rect}(-\omega)$$
$$= 2\pi\,\text{rect}(\omega)$$

We next apply scaling

$$\text{sinc}(at/2\pi) \overset{CTFT}{\Longleftrightarrow} \frac{2\pi}{a}\,\text{rect}(\omega/a)$$

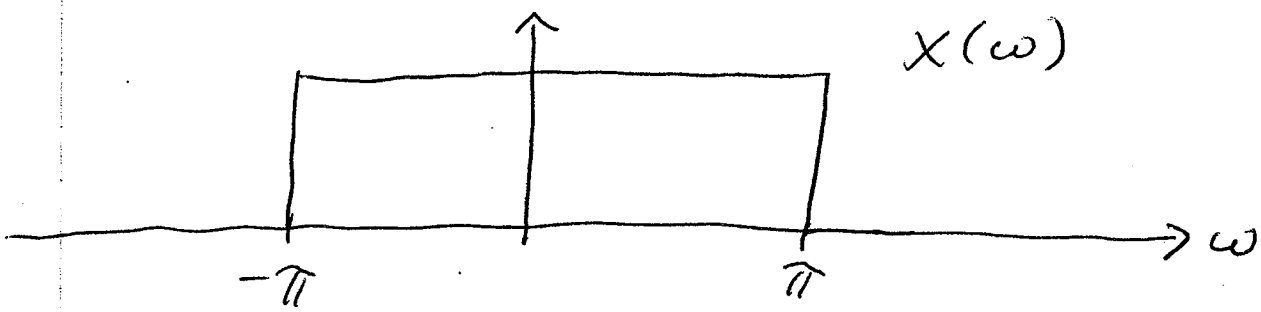
Let $a = 2\pi$

$$\text{sinc}(t) \overset{CTFT}{\Longleftrightarrow} \text{rect}(\omega/2\pi)$$

$$\boxed{\text{sinc}(t) \overset{CTFT}{\Longleftrightarrow} \text{rect}(\omega/2\pi)}$$

$$x(t) = \mathrm{sinc}(t)$$

$$X(\omega) = \mathrm{rect}(\omega/2\pi)$$



Comments

1) Only contains frequencies up to $|\omega| < \pi$

2) Contains no frequencies above $|\omega| > \pi$

3) Perfect Low Pass Filter

CTFT Pairs

$$\delta(t) \overset{CTFT}{\Longleftrightarrow} 1$$

$$\delta(t-t_0) \overset{CTFT}{\Longleftrightarrow} e^{-j\omega t_0}$$

(P3 - time shift)

$$1 \overset{CTFT}{\Longleftrightarrow} 2\pi\delta(\omega) \quad \text{(P6- Duality)}$$

$$e^{j\omega_0 t} \overset{CTFT}{\Longleftrightarrow} 2\pi\delta(\omega-\omega_0)$$

(P7 - Freq shift)

$$\cos(\omega_0 t) \overset{CTFT}{\Longleftrightarrow} \pi\left(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right)$$

(P1-linearity)

$$\sin(\omega_0 t) \overset{CTFT}{\Longleftrightarrow} -j\pi\delta(\omega-\omega_0)+j\pi\delta(\omega+\omega_0)$$

(P1- linearity)

$a>0$ $$e^{-at}u(t) \overset{CTFT}{\Longleftrightarrow} \frac{1}{j\omega+a}$$

$$e^{-a|t|} \overset{CTFT}{\Longleftrightarrow} \frac{2a}{\omega^2+a^2} \quad \text{(P1-linearity)}$$

$$\text{rect}(t) \overset{CTFT}{\Longleftrightarrow} \text{sinc}(\omega/2\pi)$$

$$\text{sinc}(t) \overset{CTFT}{\Longleftrightarrow} \text{rect}(\omega/2\pi)$$

(P6- Duality
P4- scaling)

Example

$$\frac{d^2y}{dt^2} + \alpha_1 \frac{dy}{dt} + \alpha_0 y(t) = x(t)$$

Take the Fourier transform

$$\mathcal{F}\left\{ \frac{d^2y}{dt^2} + \alpha_1 \frac{dy}{dt} + \alpha_0 y(t) = x(t) \right\}$$

$$(j\omega)^2 Y(\omega) + \alpha_1 j\omega Y(\omega) + \alpha_0 Y(\omega) = X(\omega)$$

$$X(\omega) \rightarrow \boxed{\begin{matrix} h(t) \\ H(\omega) \end{matrix}} \rightarrow Y(\omega)$$

$$Y(\omega) = H(\omega) X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \mathcal{F}\{h(t)\}$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{(j\omega)^2 + \alpha_1 j\omega + \alpha_0}$$

$$h(t) = \mathcal{F}^{-1}\left\{ \frac{1}{(j\omega)^2 + \alpha_1 j\omega + \alpha_0} \right\}$$

Important !!

P9 – Time domain convolution

$$x(t) * y(t) \overset{CTFT}{\Longleftrightarrow} X(\omega) \cdot Y(\omega)$$

$$\begin{pmatrix}\text{Time domain}\\ \text{convolution}\end{pmatrix} \overset{CTFT}{\Longleftrightarrow} \begin{pmatrix}\text{Freq. domain}\\ \text{multiplication}\end{pmatrix}$$

P10 – Time domain differentiation

Let

$$x(t) \overset{CTFT}{\Longleftrightarrow} X(\omega)$$

then

$$\frac{dx(t)}{dt} \overset{CTFT}{\Longleftrightarrow} j\omega X(\omega)$$

P11 - Parseval's Theorem

$$x(t) \overset{CTFT}{\Longleftrightarrow} X(\omega)$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

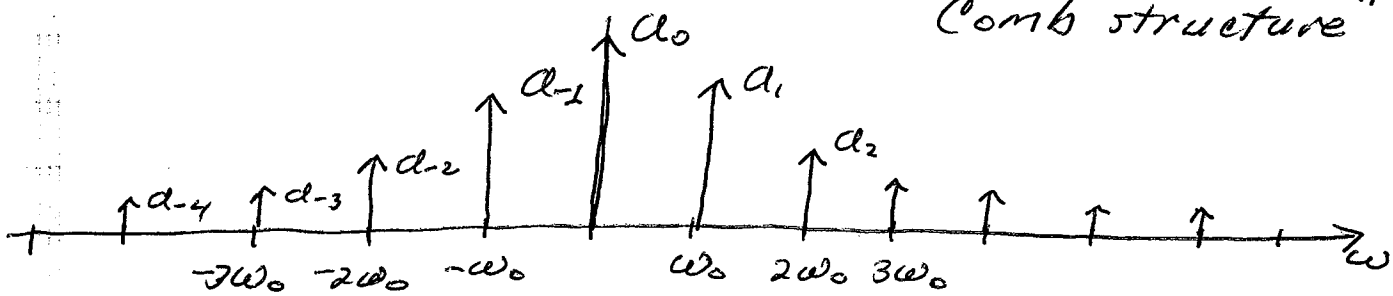
$$\begin{pmatrix} \text{Time domain} \\ \text{Energy} \end{pmatrix} = \begin{pmatrix} \text{Freq. domain} \\ \text{Energy} \end{pmatrix}$$

CTFT's of Periodic Functions

Consider the CTFT

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \, \delta(\omega - k\omega_0)$$

"Comb structure"



The inverse CTFT is given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} 2\pi a_k \, \delta(\omega - k\omega_0) \right) e^{j\omega t} \, d\omega$$

$$= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} a_k \, \delta(\omega - k\omega_0) e^{j\omega t} \, d\omega$$

$$= \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}$$

So we have

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\omega_0 t}$$

This is a Fourier Series Expansion!!

$x(t)$ is periodic!

Comments: CTFT of periodic function

1) Forms impulses in Frequency.

2) Impulses are spaced at fundamental frequency ω_0.

3) Size of impulses are proportional to Fourier series coefficients

4) Is known as a "comb"

P12 – CTFTs of periodic functions

Let $x(t)$ be a periodic function with Fourier Series expansion

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

where $\omega_0 = 2\pi/T$. Then

$$x(t) \overset{CTFT}{\Longleftrightarrow} X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

<u>Example</u>

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

We know that

$$x(t) \overset{CTFS}{\Longleftrightarrow} a_k = \frac{1}{T}$$

So by (P12) we have that

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

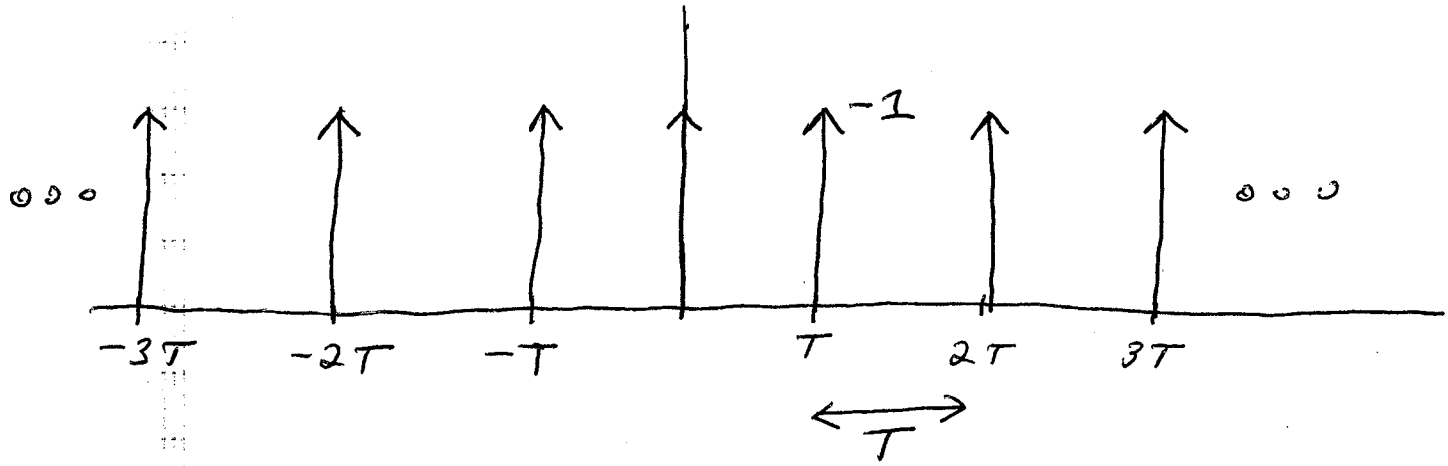
$$= \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - k\omega_0)$$

$$X(\omega) = \omega_0 \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

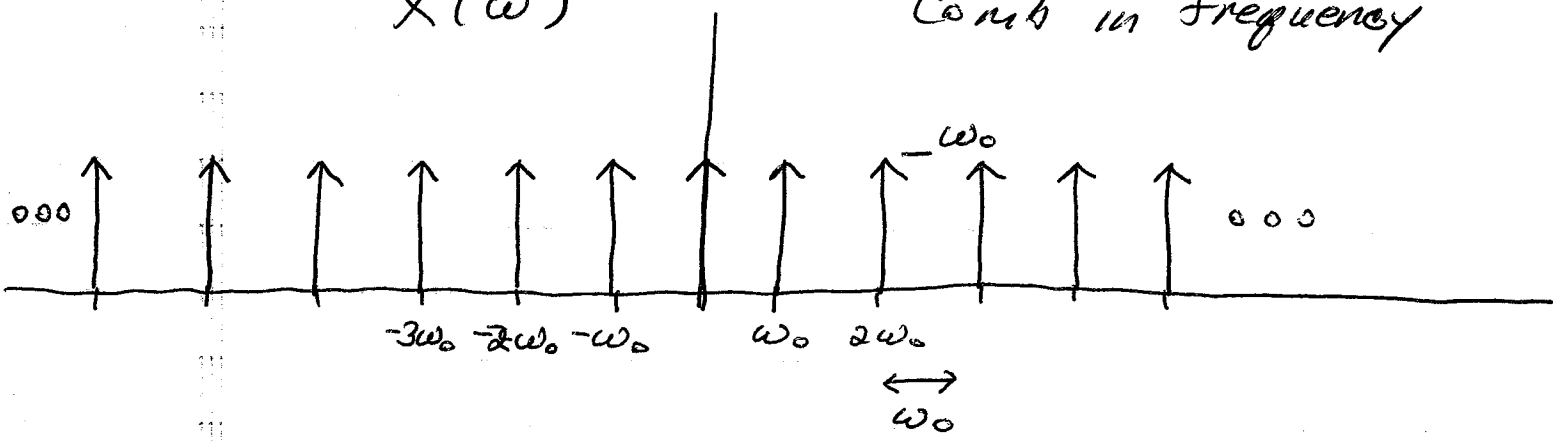
where $\omega_0 = 2\pi/T$

$$\sum_{k=-\infty}^{\infty} \delta(t - kT) \overset{CTFT}{\Longleftrightarrow} \omega_0 \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

$x(t)$ Comb in time



$X(\omega)$ Comb in frequency



Comments

1) As T grows, ω_0 decreases

2) As T decreases, ω_0 grows.

Example

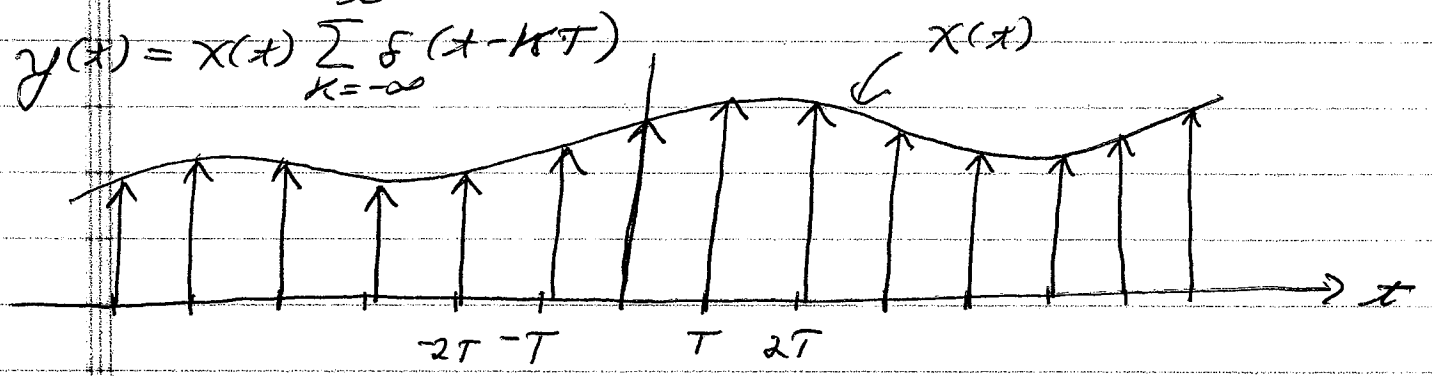
$$x(t) \overset{CTFT}{\Longleftrightarrow} X(\omega)$$

$$y(t) = x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

$$= \sum_{k=-\infty}^{\infty} x(t)\,\delta(t-kT)$$

$$= \sum_{k=-\infty}^{\infty} x(kT)\,\delta(t-kT)$$

$$y(t) = x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT)$$



$$Y(\omega) = ?$$

$$Y(\omega) = \frac{1}{2\pi} X(\omega) * \mathcal{F}\left\{ \sum_{k=-\infty}^{\infty} \delta(t-kT) \right\}$$

(P8 - Time domain multiplication)

$$= \frac{1}{2\pi} X(\omega) * \left(\omega_0 \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) \right)$$

$$\omega_0 = 2\pi/T$$

$$= \frac{\omega_0}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega) * \delta(\omega - k\omega_0)$$

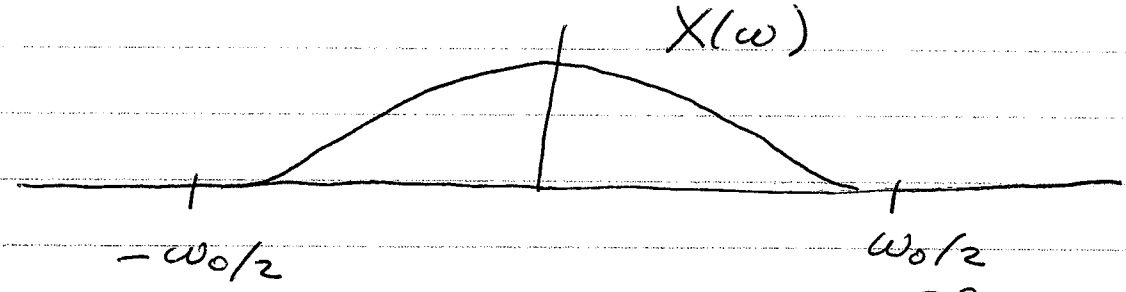
$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_0)$$

- This is a periodic replication of $X(\omega)$ with period $\omega_0 = 2\pi/T$

So we have

$$x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT) \overset{CTFT}{\Longleftrightarrow} \frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_0)$$

Assume that $X(\omega)$ is limited to $|\omega| \leq \omega_0/2$

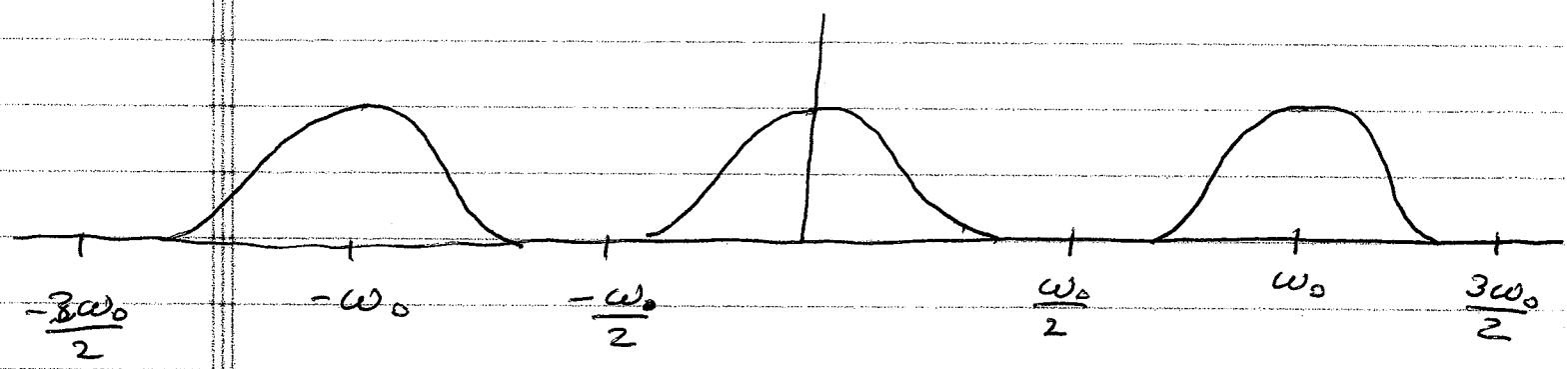


$$= \frac{2\pi}{T}\frac{1}{2}$$

$$= \frac{1}{2} \text{ fundamental}$$

then

$$\frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_0)$$



- No overlap ⇒ no loss of information about $x(t)$ ⇒ ideal sampling

Nyquist criteria

$$(\text{sampling freq}) \geq 2\left(\begin{matrix}\text{maximum}\\ \text{signal freq}\end{matrix}\right)$$

Periodic CTFT Pairs

$$\sum_{k=-\infty}^{\infty} \delta(t-kT) \overset{CTFT}{\Longleftrightarrow} \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

where $\omega_0 = 2\pi/T$

$$x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT) \overset{CTFT}{\Longleftrightarrow}$$

$$\frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_0)$$

where $\omega_0 = 2\pi/T$

By duality

$$\sum_{k=-\infty}^{\infty} x(t-kT) \overset{CTFT}{\Longleftrightarrow} X(\omega) \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

$$\begin{pmatrix} \text{replication in} \\ \text{time} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \text{comb in} \\ \text{frequency} \end{pmatrix}$$

Example:

$$\frac{d^2y}{dt^2} + 2\gamma\omega_0 \frac{dy}{dt} + \omega_0^2 y(t) = \omega_0^2 x(t)$$

$$Y(\omega)\left((j\omega)^2 + 2\gamma\omega_0 (j\omega) + \omega_0^2 \right) = \omega_0^2 X(\omega)$$

$$H(\omega) = \frac{\omega_0^2}{(j\omega)^2 + 2\gamma\omega_0 (j\omega) + \omega_0^2}$$

$$= \frac{1}{\left(j\frac{\omega}{\omega_0}\right)^2 + 2\gamma\left(j\frac{\omega}{\omega_0}\right) + 1}$$

Define $\nu = \frac{\omega}{\omega_0}$

$$c_1 = -\gamma + \sqrt{\gamma^2 - 1}$$
$$c_2 = -\gamma - \sqrt{\gamma^2 - 1}$$

$$H(\omega) = \frac{1}{(j\nu - c_1)(j\nu - c_2)}$$

$$\frac{1}{(j\nu - c_1)(j\nu - c_2)} = \frac{A_1}{(j\nu - c_1)} + \frac{A_2}{(j\nu - c_2)}$$

$$1 = A_1 (j\nu - c_2) + A_2 (j\nu - c_1)$$

$$0 = A_1 (j\nu) + A_2 (j\nu) \Rightarrow A_1 = -A_2$$

$$1 = -A_1 c_2 - A_2 c_1$$

$$1 = -A_1 c_2 + A_1 c_1 \Rightarrow A_1 = \frac{1}{c_1 - c_2}$$

$$A_2 = \frac{1}{c_2 - c_1}$$

$$H(\omega) = \frac{1}{(c_1 - c_2)} \frac{1}{j\nu - c_1} + \frac{1}{(c_2 - c_1)} \frac{1}{(j\nu - c_2)}$$

$$= \frac{\omega_0}{c_1 - c_2} \frac{1}{j\omega - c_1\omega_0} + \frac{\omega_0}{c_2 - c_1} \frac{1}{j\omega - c_2\omega_0}$$

$$h(t) = \frac{\omega_0}{c_1 - c_2} e^{c_1\omega_0 t} u(t) + \frac{\omega_0}{c_2 - c_1} e^{c_2\omega_0 t} u(t)$$

What if $\left.\begin{array}{l} x(t) = e^{-2t}u(t) \\ \omega_0 = 1 \\ c_1 = -1 \\ c_2 = -2 \end{array}\right\}$ What is $y(t)$?

$$A_1 = \frac{\omega_0}{c_1 - c_2} = -1 \qquad A_2 = \frac{\omega_0}{c_2 - c_1} = 1$$

$$h(t) = -1e^{-t} + 1e^{-2t} = e^{-2t} - e^{-t}$$

$$H(\omega) = \frac{-1}{j\omega + 1} + \frac{1}{j\omega + 2}$$

$$= \frac{1}{(j\omega)^2 + 3(j\omega) + 2}$$

$$Y(\omega) = H(\omega)\, X(\omega)$$

$$X(\omega) = \frac{1}{j\omega + 2} = \mathcal{F}\{e^{-2t}\}$$

$$Y(\omega) = \left(\frac{-1}{(j\omega + 1)} + \frac{1}{j\omega + 2} \right) \frac{1}{j\omega + 2}$$

$$= \frac{-1}{(j\omega+1)(j\omega+2)} + \frac{1}{(j\omega+2)^2}$$

$$= \frac{A_3}{j\omega + 1} + \frac{A_4}{j\omega + 2} + \frac{1}{(j\omega+2)^2}$$

$$-1 = A_3 (j\omega + 2) + A_4 (j\omega + 1)$$

$$A_3 = -A_4$$

$$-1 = 2A_3 + A_4$$

$$= 2A_3 - A_3 \Rightarrow A_3 = -1$$

$$A_4 = 1$$

$$Y(\omega) = \frac{-1}{j\omega + 1} + \frac{1}{j\omega + 2} + \frac{1}{(j\omega + 2)^2}$$

$$y(t) = -e^{-t}u(t) + e^{-2t}u(t) + \mathcal{F}^{-1}\left\{ \frac{1}{(j\omega+2)^2} \right\}$$

?

$$\mathcal{F}^{-1}\left\{ \frac{1}{(j\omega+2)^2} \right\} = \mathcal{F}^{-1}\left\{ \frac{1}{(j\omega+2)} \right\} * \mathcal{F}^{-1}\left\{ \frac{1}{j\omega+2} \right\}$$

$$= \left(e^{-2t}u(t) \right) * \left(e^{-2t}u(t) \right)$$

$$= t e^{-2t} u(t)$$

$$\boxed{\mathcal{F}^{-1}\left\{\frac{1}{(j\omega+a)^n}\right\} = \frac{t^{n-1}e^{-at}}{(n-1)!}\,u(t)}$$

$$y(t) = \left(-e^{-t} + e^{-2t} + te^{-2t}\right)u(t)$$

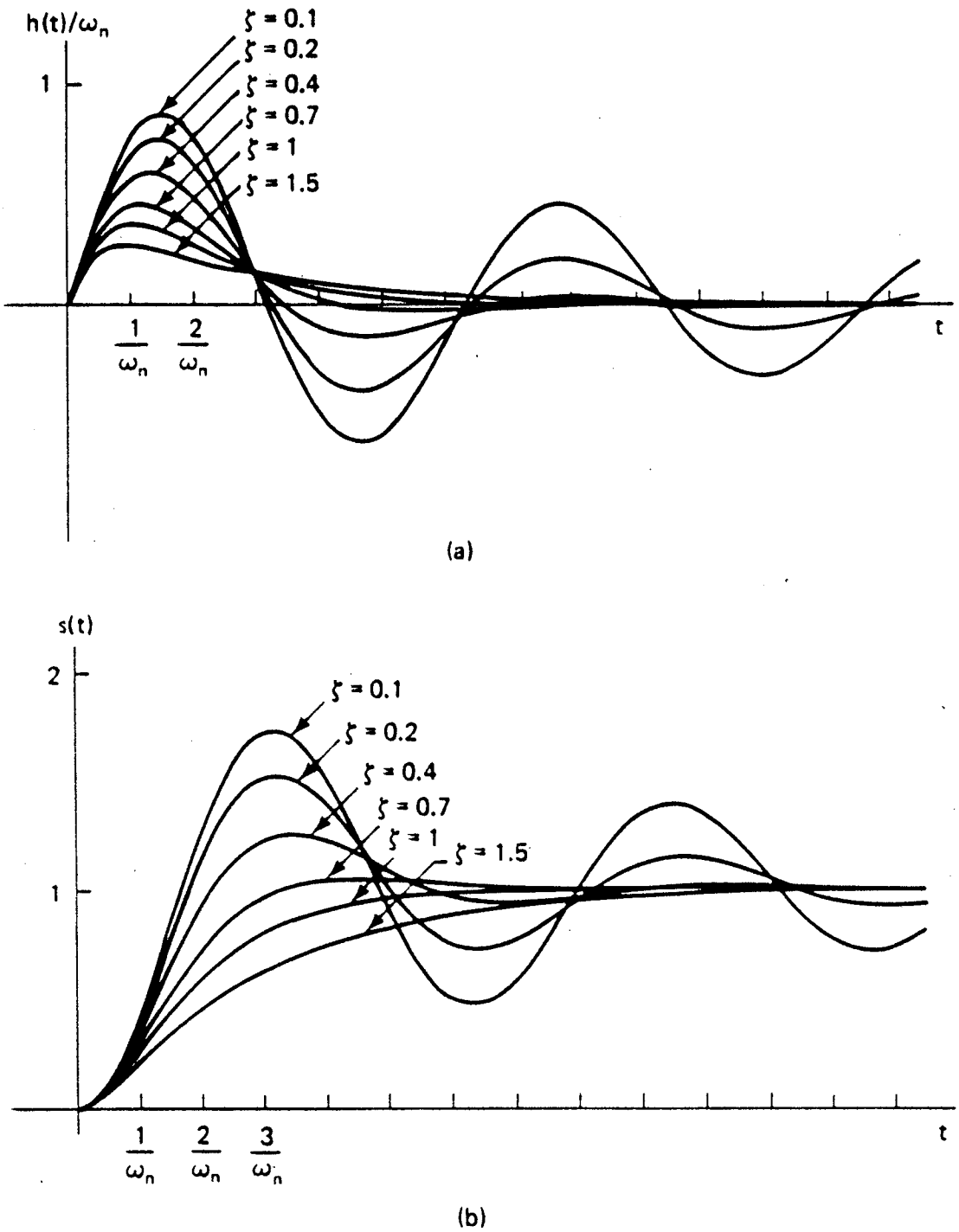


Figure 4.42 (a) Impulse responses and (b) step responses for second-order systems with different values of the damping ratio ζ.

General case

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$\mathcal{F}\{\ \}$

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} \quad \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

$$\text{rational function} = \frac{\text{polynomial of } (j\omega)}{\text{polynomial of } (j\omega)}$$

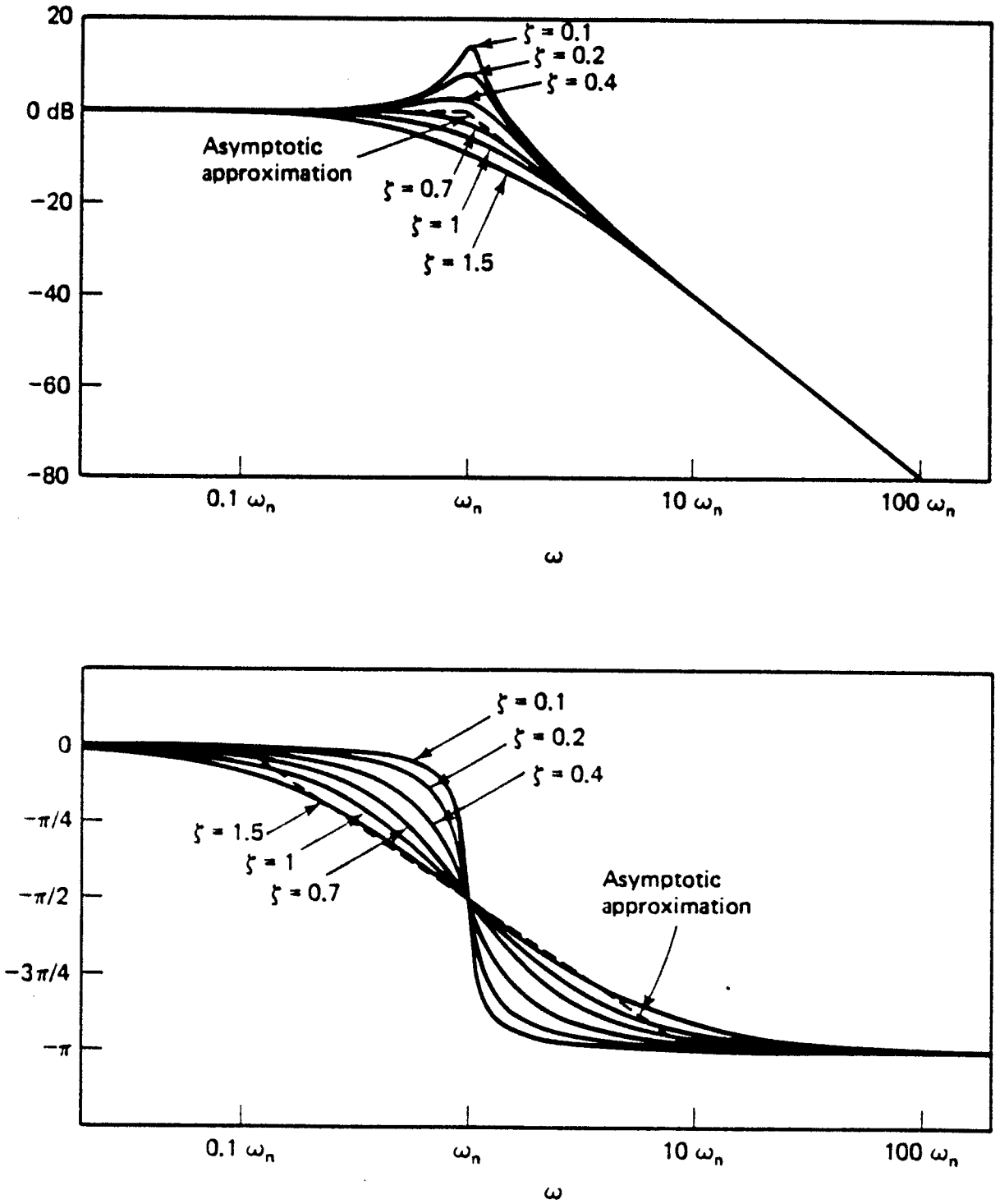


Figure 4.43 Bode plots for second-order systems with several different values of damping ratio ζ.