Lecture 10

Z-transform

- Z-T

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

+ Region of convergence (ROC)

- Inverse

$$x(n) = \frac{1}{2\pi j} \oint_C X(z)\, z^{n-1}\, dz$$

C in the ROC

- The ROC of $X(z)$ is the set of z's for which the transform converges ($X(z) < \infty$).

Example 1 : Right sided

$$x(n) = a^n u(n) \quad \left(\begin{array}{l} a \text{ is any complex} \\ \text{number} \end{array}\right)$$

$$X(z) = \sum_n a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$ROC = \left|\frac{a}{z}\right| < 1 \Rightarrow |z| > |a|$$

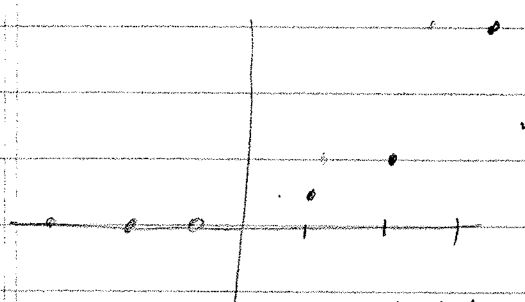
$$X(z) = \frac{1}{1-\left(\frac{a}{z}\right)}$$

$$= \frac{z}{z-a}$$

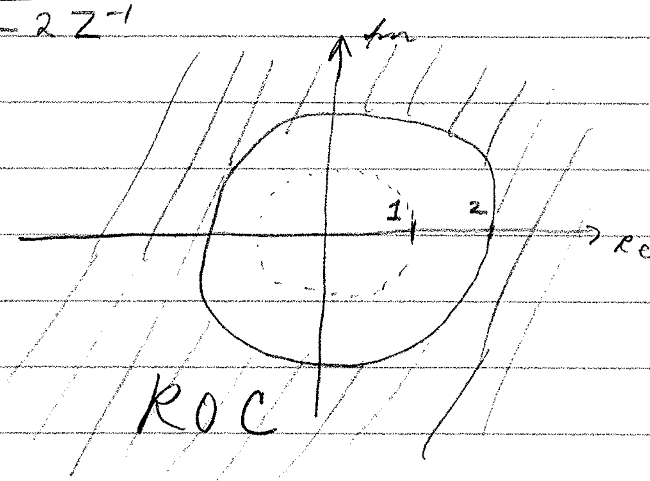
$$= \frac{1}{1-az^{-1}} \qquad ROC = |z| > |a|$$

Let $a = 2 \quad x(n) = 2^n u(n)$

$$X(z) = \frac{1}{1-2z^{-1}}$$



unstable



Note! DTFT of $x(n) = 2^n u(n)$ does not exist. Why?

$$X(e^{j\omega}) = X(z)\Big|_{z=e^{j\omega}}$$

but $e^{j\omega} \notin ROC$

Conclusion:

DTFT exists $\Leftrightarrow$ (unit circle is in ROC)

Example 2: Left-sided sequence

$$x(n) = -a^n u(-n-1)$$

$$X(z) = -\sum_n a^n u(n-1) z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} \left(\frac{a}{z}\right)^n$$

$$= -\sum_{n=1}^{\infty} \left(\frac{a}{z}\right)^{-n}$$

$$= -\left(\frac{z}{a}\right) \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^n$$

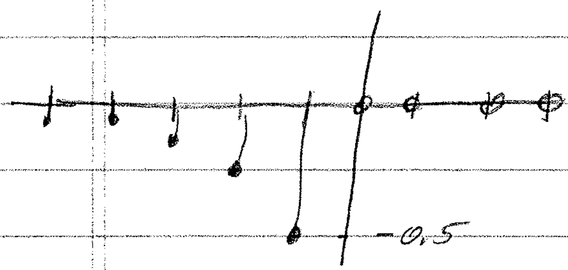
$$ROC = \left|\frac{z}{a}\right| < 1 \quad |z| < |a|$$

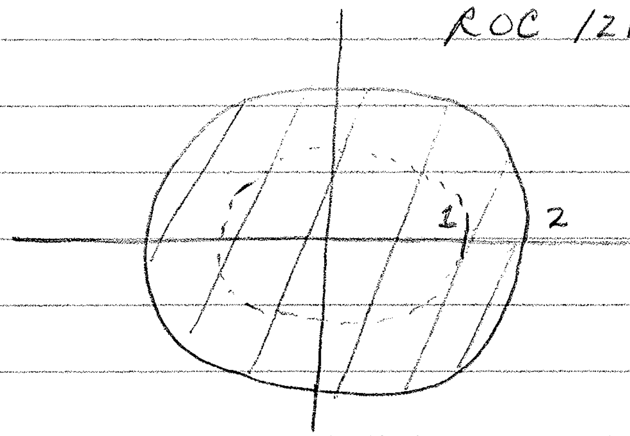
$$X(z) = -\frac{z}{a} \frac{1}{1-\frac{z}{a}} = \frac{z}{z-a}$$

$$= \frac{1}{1-az^{-1}}$$

Let $a = 2$ $\quad x(n) = -2^n(-n-1)$

ROC $|z| < 2$





Note: $2^n u(n)$ and $-2^n(-n-1)$ have the same function as a Z-T but different ROC.

Example 3 Two-Sided Sequence

$$x(n) = a^{-|n|}$$
$$= a^n u(-n-1) + \left(\frac{1}{a}\right)^n u(n)$$

$$X(z) = \frac{-1}{1-az^{-1}} + \frac{1}{1-\frac{1}{a}z^{-1}}$$

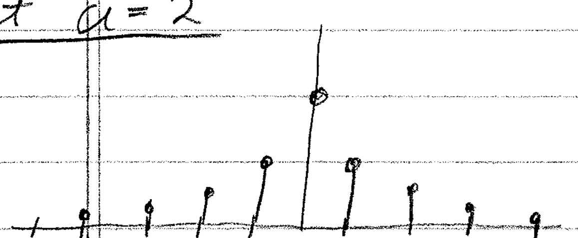
$$ROC = \overset{ROC}{\{|z|<|a|\}} \cap \overset{ROC}{\{|z|>\tfrac{1}{|a|}\}}$$

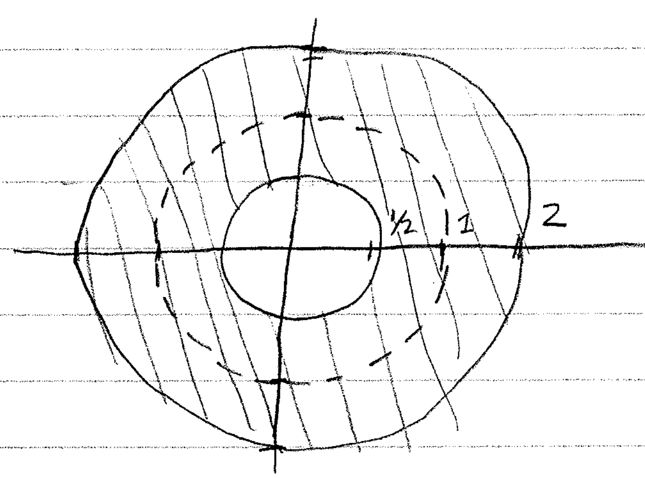
$$X(z) = \frac{-\left(1-\frac{1}{a}z^{-1}\right)+\left(1-az^{-1}\right)}{\left(1-az^{-1}\right)\left(1-\frac{1}{a}z^{-1}\right)}$$

$$= \frac{\left(\frac{1}{a}-a\right)z^{-1}}{\left(1-az^{-1}\right)\left(1-\frac{1}{a}z^{-1}\right)}$$

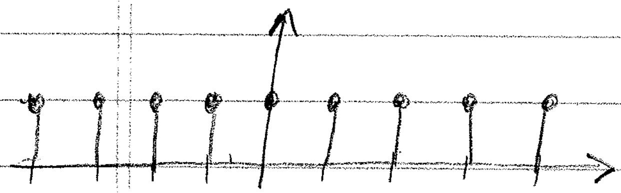
$$= \frac{\left(\frac{1}{a}-a\right)z}{(z-a)\left(z-\frac{1}{a}\right)}$$

Let $a=2$





Let $\underline{a = 1}$



$$ROC = \{|z|<1\} \cap \{|z|>1\} = \phi$$

Z-T does not exist!

General Z-T ROC Properties:

1. right-sided sequence

a) ROC $= \{|z| > a\}$ for some a

b) sequence is bounded $\iff a < 1$

2. left-sided sequence

a) ROC $= \{|z| < b\}$ for some b

b) sequence is bounded $\iff b > 1$

3. Two sided

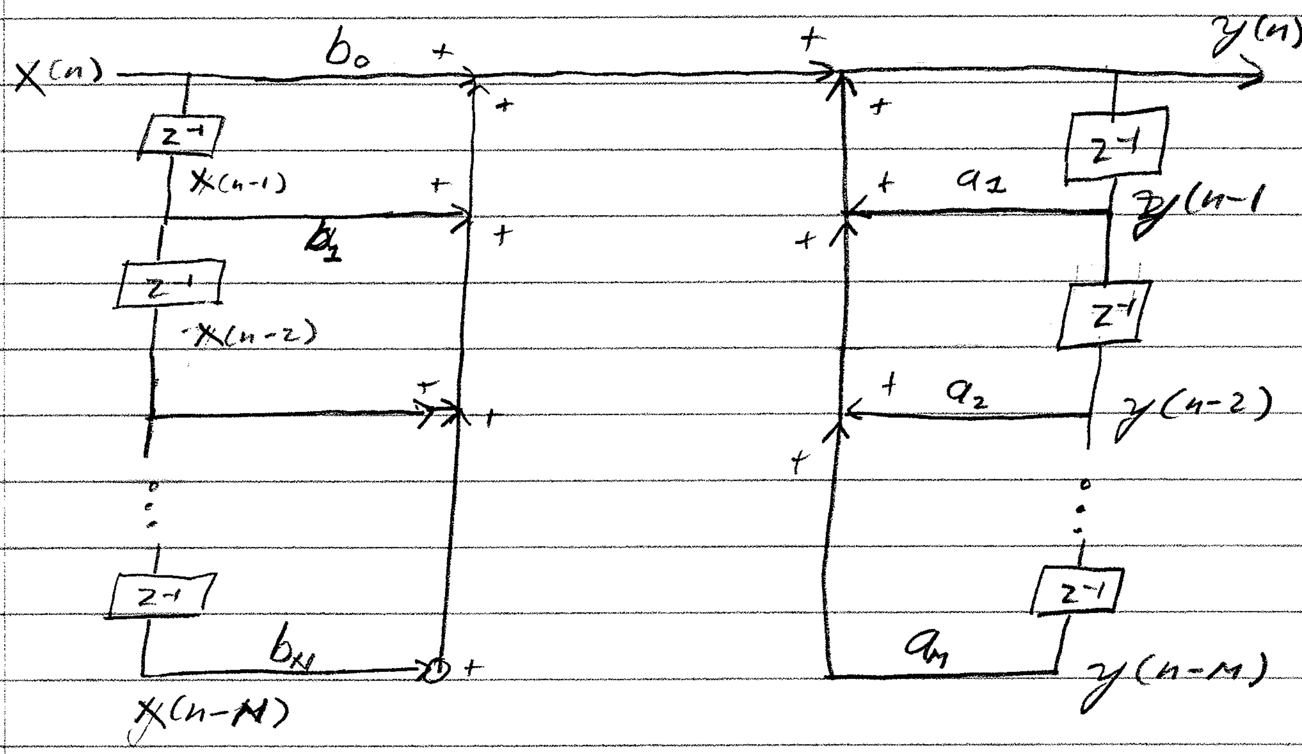
a) ROC $= \{a < |z| < b\}$ (annulus)

b) sequence is bounded $\iff a < 1 < b$ $\iff$ ROC contains unit circle

General Difference Equation

$$y(n) = \sum_{k=1}^{M} a_k\, y(n-k) + \sum_{\ell=0}^{N} b_\ell\, x(n-\ell)$$

Flow diagram



Z-Transform

$$Y(z) = \sum_{k=1}^{M} a_k\, Y(z) z^{-k} + \sum_{\ell=0}^{N} b_\ell\, X(z) z^{-\ell}$$

$$Y(z)\left(1 - \sum_{k=1}^{M} a_k z^{-k}\right) = X(z) \sum_{\ell=0}^{N} b_\ell z^{-\ell}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{\ell=0}^{N} b_\ell z^{-\ell}}{1 - \sum_{k=1}^{M} a_k z^{-k}}$$

$$= b_0 z^{M-N} \frac{\sum_{\ell=0}^{N} \left(\frac{b_\ell}{b_0}\right) z^{N-\ell}}{z^M - \sum_{k=1}^{M} a_k z^{M-k}}$$

- By the fund. theorem of algebra:

$$H(z) = b_0 z^{M-N} \frac{\prod_{l=1}^{N} (z - z_l)}{\prod_{k=1}^{M} (z - p_k)}$$

z_k – zeros
p_k – poles

- Magnitude and phase of DTFT

$$|H(e^{j\omega})| = b_0 \frac{\prod_{l=1}^{N} |e^{j\omega} - z_l|}{\prod_{k=1}^{M} |e^{j\omega} - p_k|}$$

$$\angle H(e^{j\omega}) = \omega(M-N) + \sum_{l=1}^{M} \angle (e^{j\omega} - z_l) - \sum_{k=1}^{N} \angle (e^{j\omega} - p_k)$$

Relationship of ROC to Causality and Stability

- For a causal system

$$\mathrm{ROC} = \{\, |z| > |p_{max}| \,\}$$

  where p_{max} is pole with largest eigenvalue

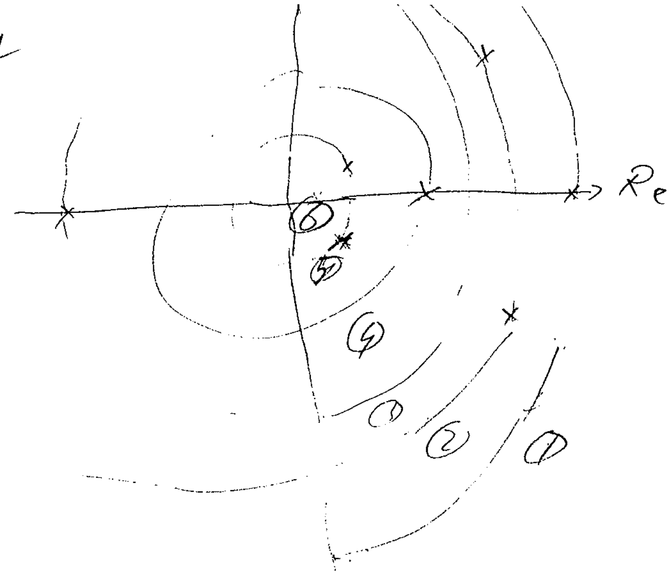
- Stability

A system is stabile $\Leftrightarrow$ ROC included unit circle

- Conclusion

$$\boxed{\text{A causal system is stable} \Leftrightarrow \text{all poles lie within the unit circle}}$$

Examples



(Real poles or zeros don't have to be in pairs)

Note: ① No ROC ~~must~~ can include a pole (because poles blow up)

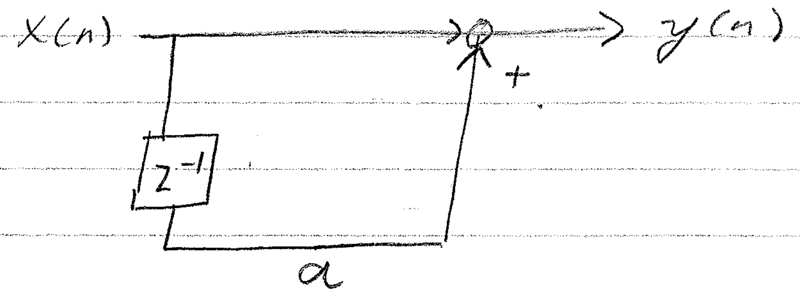
② ~~Each~~ There are 6 possible ROCs

③ Each region has its own impulse response!

④ Only region ① is causal.

Example: FIR Filter

$$y(n) = x(n) + a\,x(n-1)$$



$$Y(z) = X(z)\left(1 + a z^{-1}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 + a z^{-1} \cdot \frac{z}{z}$$

$$= \frac{z + a}{z}$$

zero at $-a$
pole at 0 $\Big\}$ $\Rightarrow$ always stable

• Find an inverse filter $G(z)$

$$y(n) \longrightarrow \boxed{G(z)} \longrightarrow x(n)$$

$$x(n) = G(z)\, y(n)$$
$$= G(z)\, H(z)\, x(n)$$

$$G(z) = \frac{1}{H(z)} = \frac{1}{1 + az^{-1}} \quad |z| > |a|$$

$$g(n) = (-a)^n u(n)$$

$G(z)$ has:

$\left.\begin{array}{l}\text{zero at } 0 \\ \text{pole at } -a\end{array}\right\} \Rightarrow$ only stable if $|a| < 1$

<u>Let $a = 1/2$</u>

$g(n) = (-1/2)^n u(n)$ <u>stable</u>

<u>Let $a = 2$</u>

$g(n) = (-2)^n u(n)$ <u>unstable</u>

EE 438 Z-transform Example

Determine the frequency and impulse response of the following **causal** system.

$$y(n) = -\frac{1}{2}y(n-2) + x(n) + x(n-1)$$

Analysis:

$$Y(z) = \frac{1}{2}z^{-2}Y(z) + X(z) + z^{-1}X(z)$$

$$Y(z)\left(1 + \frac{1}{2}z^{-2}\right) = X(z)\left(1 + z^{-1}\right)$$

$$\begin{aligned} H(z) &= \frac{1 + z^{-1}}{1 + \frac{1}{2}z^{-2}} \\ &= \frac{z(z+1)}{z^2 + \frac{1}{2}} \\ &= \frac{z(z+1)}{\left(z + j\frac{1}{\sqrt{2}}\right)\left(z - j\frac{1}{\sqrt{2}}\right)} \end{aligned}$$

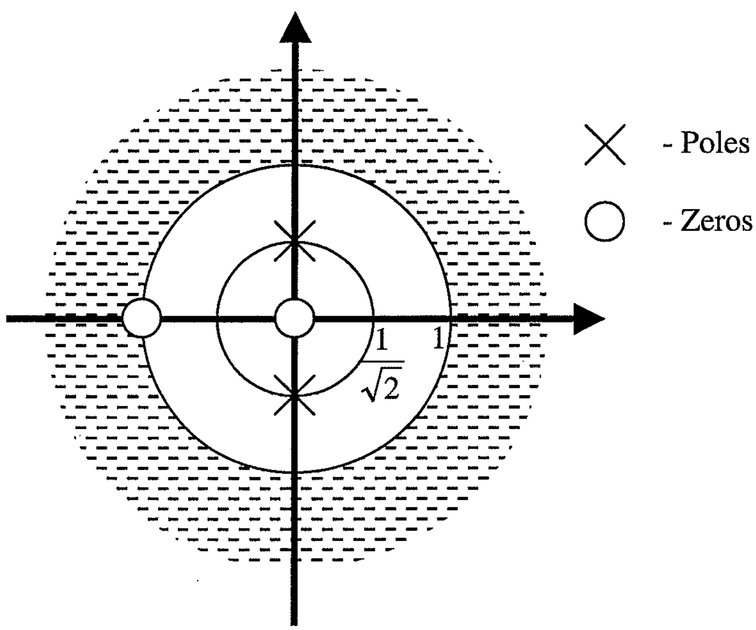
Then the region of convergence is:

Causal => $h(n)$ is right sided

$\Rightarrow$ ROC $= \{|z| > b\}$ where $b = \max_k |p_k| = \frac{1}{\sqrt{2}}$

$\Rightarrow$ ROC $= \{|z| > 1/\sqrt{2}\}$

Pole-zero diagram:



Compute impulse response

$$h(n) = Z^{-1}\{H(z)\}$$

Use partial fraction expansion (see appendix of Oppenheim, Willsky with Young)

$$\frac{H(z)}{z} = \frac{z+1}{\left(z + j\frac{1}{\sqrt{2}}\right)\left(z - j\frac{1}{\sqrt{2}}\right)} = \frac{A}{z + j\frac{1}{\sqrt{2}}} + \frac{B}{z - j\frac{1}{\sqrt{2}}}$$

$$\begin{aligned} A &= \left.\frac{z+1}{z - j/\sqrt{2}}\right|_{z=-j/\sqrt{2}} \\ &= \frac{1 - j/\sqrt{2}}{-j\frac{2}{\sqrt{2}}} \\ &= \frac{1}{2} + j\frac{\sqrt{2}}{2} \\ &= \frac{1}{2} + j\frac{1}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} B &= \left.\frac{z+1}{z + j/\sqrt{2}}\right|_{z=j/\sqrt{2}} \\ &= \frac{1 + j/\sqrt{2}}{j\frac{2}{\sqrt{2}}} \\ &= \frac{1}{2} - j\frac{\sqrt{2}}{2} \\ &= \frac{1}{2} - j\frac{1}{\sqrt{2}} \end{aligned}$$

Notice that $A = B^*$ because $-j/\sqrt{2}$ and $+j/\sqrt{2}$ are complex conjugate pole pairs.

$$\frac{H(z)}{z} = \frac{z\left(\frac{1}{2} + j\frac{1}{\sqrt{2}}\right)}{z + j\frac{1}{\sqrt{2}}} + \frac{z\left(\frac{1}{2} - j\frac{1}{\sqrt{2}}\right)}{z - j\frac{1}{\sqrt{2}}}$$

Since the ROC = $|z| > \frac{1}{\sqrt{2}}$

$$h(n) = \left(\frac{1}{2} + j\frac{1}{\sqrt{2}}\right)\left(-\frac{j}{\sqrt{2}}\right)^n u(n) + \left(\frac{1}{2} - j\frac{1}{\sqrt{2}}\right)\left(+\frac{j}{\sqrt{2}}\right)^n u(n)$$

Since $j = e^{j\pi/2}$ and $\left(\frac{1}{2} + j\frac{1}{\sqrt{2}}\right) = \frac{\sqrt{3}}{2} e^{j\tan^{-1}\sqrt{2}}$

$$\begin{aligned} h(n) &= \frac{\sqrt{3}}{2} e^{j\tan^{-1}\sqrt{2}} \left(\frac{1}{\sqrt{2}}\right)^n e^{-j\pi n/2} u(n) + \frac{\sqrt{3}}{2} e^{-j\tan^{-1}\sqrt{2}} \left(\frac{1}{\sqrt{2}}\right)^n e^{j\pi n/2} u(n) \\ &= \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{2}}\right)^n 2\cos\left(\frac{\pi}{2}n - \tan^{-1}\sqrt{2}\right) u(n) \end{aligned}$$

$$h(n) = \sqrt{3}\left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{\pi}{2}n - \tan^{-1}\sqrt{2}\right) u(n)$$