Time Domain Analysis of Linear Systems

1. Discrete time systems

Consider the Linear Time Invariant (LTI) system

$$y_n = T[x_n]$$

- The output y_n is a function of the complete input x_n

- We know from property 2 of DT impulses

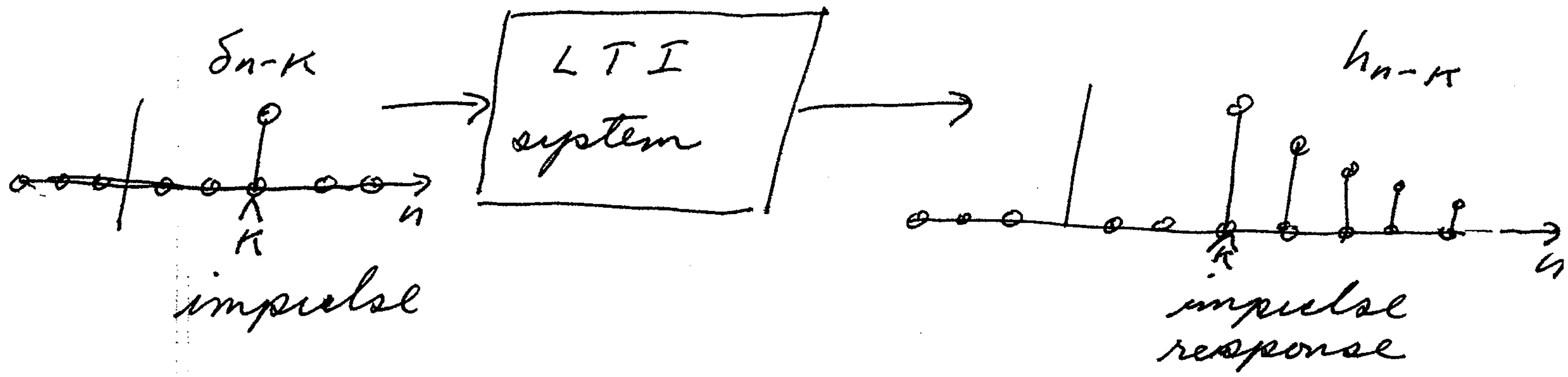
$$x_n = \sum_{k=-\infty}^{\infty} x_k \delta_{n-k}$$

Therefore

$$\begin{aligned}
y_n &= T[x_n] \\
&= T\left[\sum_{k=-\infty}^{\infty} x_k \delta_{n-k}\right] \\
&= \sum_{k=-\infty}^{\infty} T[x_k \delta_{n-k}] \\
&= \sum_{k=-\infty}^{\infty} x_k T[\delta_{n-k}]
\end{aligned}$$

What is $T[\delta_{n-k}]$?

The response of S to an impulse at time k.



$$h_n \triangleq T[\delta_n]$$

h_n is called the <u>impulse response</u> of the system S.

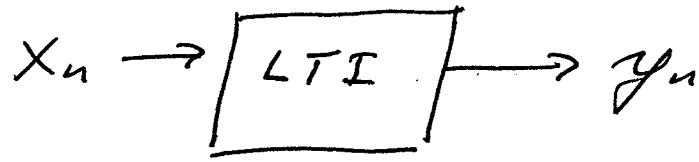
Since S is LTI $\Rightarrow$ $h_{n-k} = T[\delta_{n-k}]$

$$y_n = \sum_{k=-\infty}^{\infty} x_k T[\delta_{n-k}]$$

$$\boxed{y_n = \sum_{k=-\infty}^{\infty} x_k h_{n-k}}$$

By measuring h_n, we know the response of the system to <u>any</u> input!

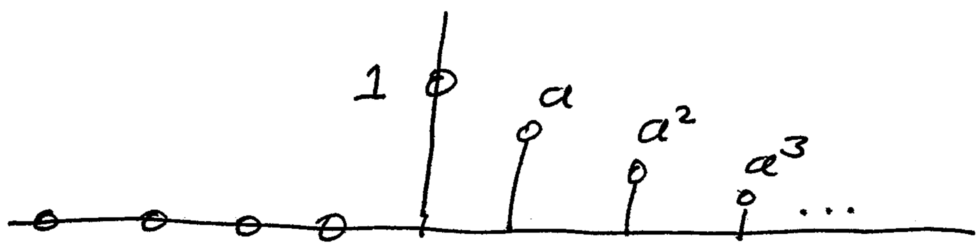
Example



$$y_n = T[x_n]$$

$$h_n = T[\delta_n] = a^n u_n \cancel{(n)}$$

where $|a| < 1$



What is the output when $x_n = u_n$?
(step response)

$$y_n = \sum_{k=-\infty}^{\infty} x_k h_{n-k}$$

$$= \sum_{k=-\infty}^{\infty} u_k \left(a^{n-k} u_{n-k} \right)$$

$$= \begin{cases} \sum_{k=0}^{n} a^{n-k} & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

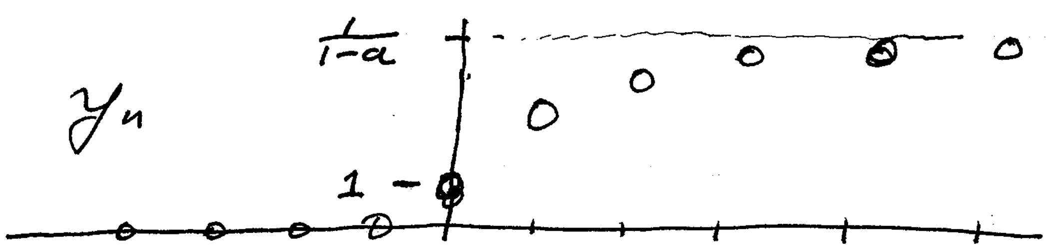
$$= u_n \left\{ \sum_{k=0}^{n} a^{n-k} \right\} \qquad \ell = n-k$$

$$= u_n \sum_{\ell=0}^{n} a^{\ell}$$

$$= u_n \left(\sum_{\ell=0}^{\infty} a^{\ell} - \sum_{\ell=n+1}^{\infty} a^{\ell} \right)$$

$$= u_n \left(\frac{1}{1-a} - a^{n+1} \sum_{\ell=0}^{\infty} a^{\ell} \right)$$

$$= u_n \left(\frac{1-a^{n+1}}{1-a} \right)$$



Remember

$$\boxed{\sum_{\ell=0}^{n-1} a^{\ell} = \frac{1-a^n}{1-a}}$$

We will use this a <u>lot</u>!

The <u>Convolution</u> of any two functions, x_n h_n is given by

$$x_n * h_n = \sum_{k=-\infty}^{\infty} x_k h_{n-k}$$

$$x_n \longrightarrow \boxed{h_n} \longrightarrow y_n = x_n * h_n$$

property 1) $x_n * h_n = h_n * x_n$

$$x_n * h_n = \sum_{k=-\infty}^{\infty} x_k h_{n-k}$$

$$= \sum_{u=-\infty} x_{n-u} h_u \qquad u = n-k$$

$$\qquad\qquad\qquad k = n-u$$

$$= \sum_{u=-\infty}^{\infty} h_u x_{n-u}$$

$$= h_n * x_n$$

property 2) $(x_n * z_n) * h_n = x_n * (z_n * h_n)$

$$\left(\sum_{k=-\infty}^{\infty} x_k z_{n-k}\right) * h_n$$

$$= \sum_{u=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_k z_{u-k} h_{n-u}$$

$$= \sum_{k=-\infty}^{\infty} x_k \sum_{u=-\infty}^{\infty} z_{u-k} h_{n-u} \qquad l = u-k$$

$$\qquad\qquad\qquad u = l+k$$

$$= \sum_{k=-\infty}^{\infty} x_k \sum_{l=-\infty}^{\infty} z_l h_{(n-k)-l}$$

$$= \sum_{k=-\infty}^{\infty} x_k \, (z_n * h_n)_{n-k}$$

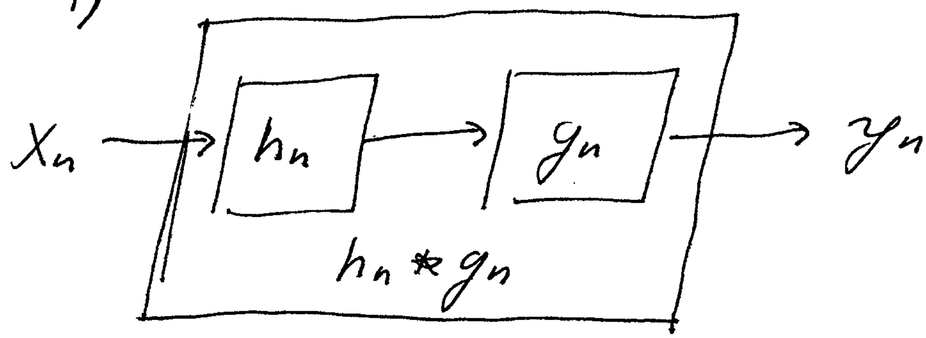
$$= \quad x_n * (z_n * h_n)$$

property 3)

$$x_n * h_n + x_n * g_n = x_n * (h_n + g_n)$$

Examples

1)

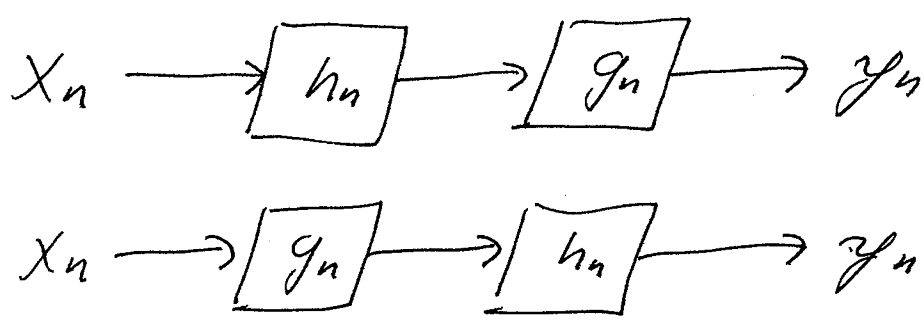


$$y_n = (x_n * h_n) * g_n$$

$$= x_n * (h_n * g_n)$$

The series connection of two LTI systems is an LTI system.

2)



$$y_n = (x_n * h_n) * g_n$$

$$= x_n * (h_n * g_n)$$

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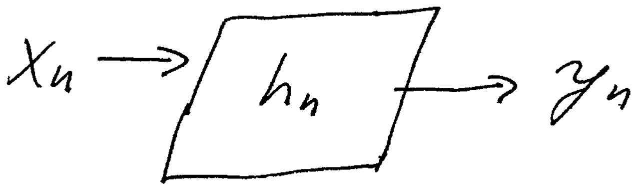
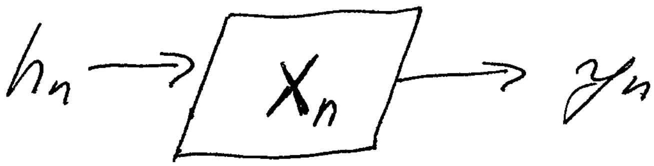
$$= (x_n * g_n) * h_n$$

The order of LTI systems can be exchanged

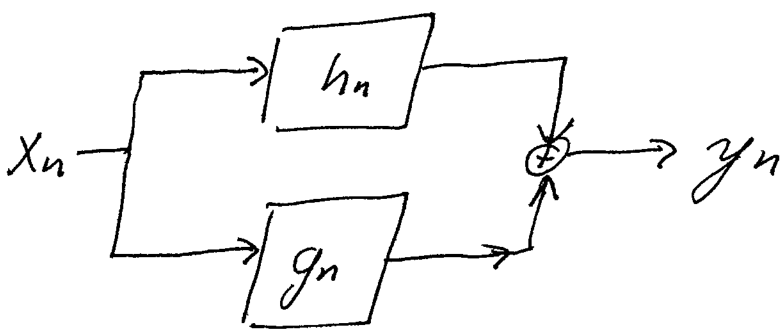
3)

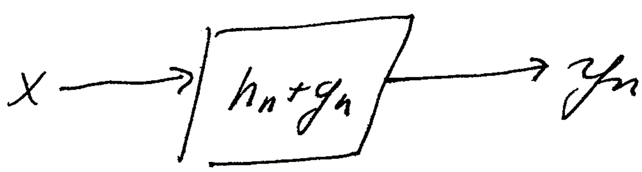




$$y_n = x_n * h_n = h_n * x_n \,!$$

property 3) $x_n * h_n + x_n * g_n = x_n * (h_n + g_n)$

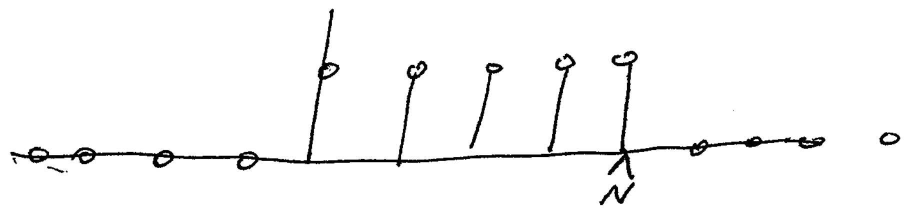




Example:

$$y_n = x_n * h_n$$

$$h_n = \begin{cases} 1 & 0 \le n \le N \\ 0 & \text{otherwise} \end{cases}$$



$$y_n = \sum_{K=-\infty}^{\infty} X_K h_{n-K} = X * h$$

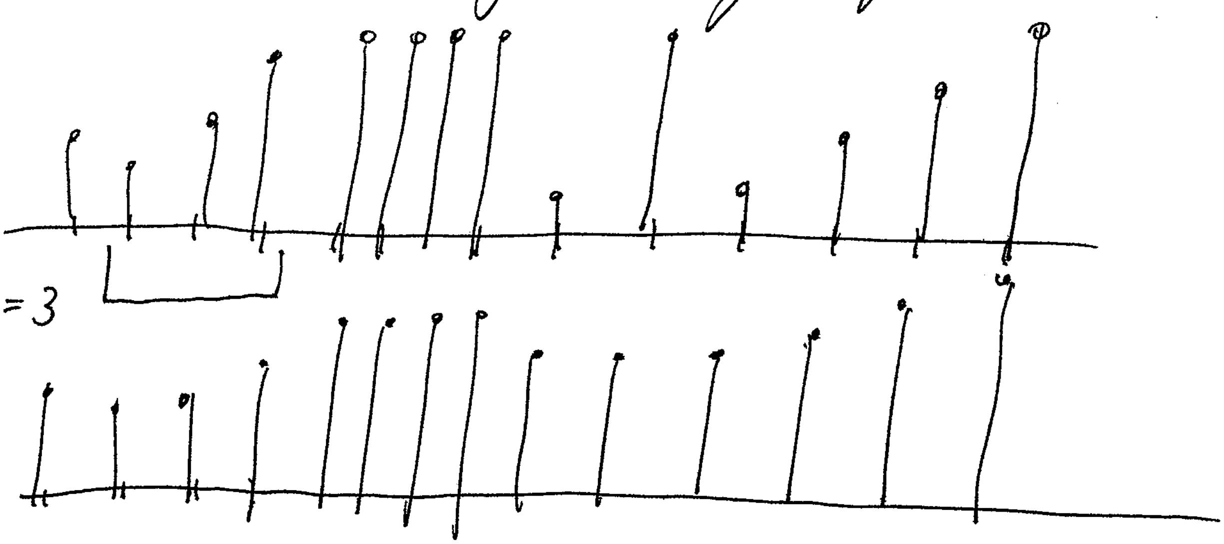
$$= \sum_{K=-\infty}^{\infty} h_K X_{n-K} = h * X$$

$$= \sum_{K=0}^{N} h_K X_{n-K}$$

$$y_n = \sum_{K=0}^{N} X_{n-K} = \sum_{K=n-N}^{n} X_K$$

moving average filter

$N = 3$



$$h_n = u_n - u_{n-N-1}$$

$$y_n = x_n * h_n = x_n * u_n + x_n * (-u_{n-N-1})$$

$$x_n * u_n = \sum_{k=-\infty}^{\infty} x_k \, u_{n-k}$$

$$= \sum_{k=-\infty}^{n} x_k \quad \leftarrow \text{integrator}$$

summer

$$x_n * (-u_{n-N-1}) = -\left(x_n * u_n\right)_{n-N-1}$$

$$= -\sum_{k=-\infty}^{n-N-1} x_k$$

$$y_n = \sum_{k=-\infty}^{n} x_k - \sum_{k=-\infty}^{n-N-1} x_k$$

$$= \sum_{k=n-N}^{n} x_k$$

Let T be a LTI continuous time system

$$y(t) = T[x(t)]$$

$$= T\Big[\underbrace{\int_{-\infty}^{\infty} x(\lambda)\,\delta(t-\lambda)\,d\lambda}_{\text{this is an infinite sum of signals}}\Big]$$

superposition $= \int_{-\infty}^{\infty} T[x(\lambda)\,\delta(t-\lambda)]\,d\lambda$

Homogeneity $= \int_{-\infty}^{\infty} x(\lambda)\,[\delta(t-\lambda)]\,d\lambda$

$T[\delta(t-\lambda)]$ – continuous time impulse response

$= h(t-\lambda)$

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\lambda)\,h(t-\lambda)\,d\lambda}$$

continuous time convolution integral

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)\,h(t-\lambda)\,d\lambda$$

Properties of CT Convolution

property 1)

$$x(t) * h(t) = h(t) * x(t)$$

property 2)

$$(x(t) * h(t)) * g(t) = x(t) * (h(t) * g(t))$$

property 3)

$$x(t) * (h(t) + g(t)) = x(t) * h(t) + x(t) * g(t)$$

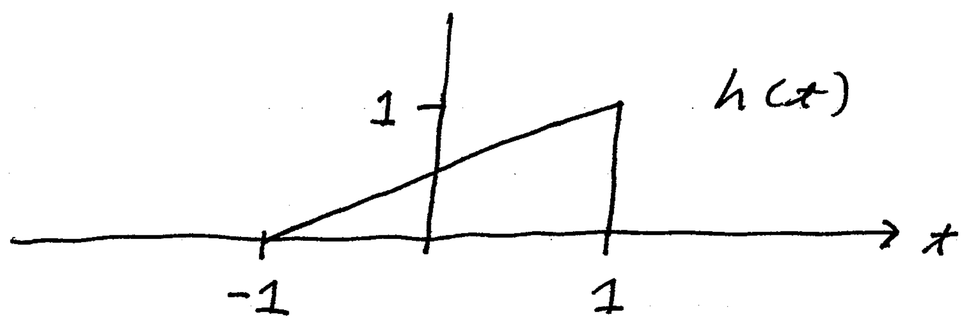
Flip and Shift

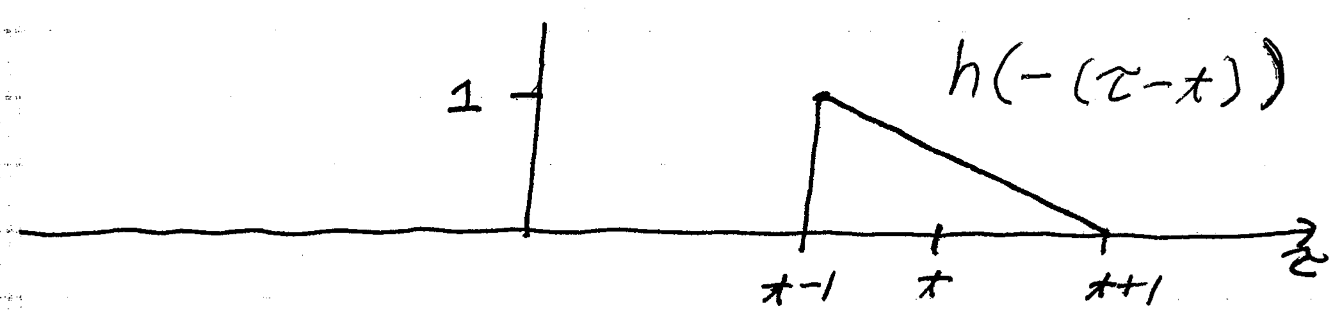
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Think of $x(\tau)$ and $h(t-\tau)$ as functions of $\underline{\tau}$

$$h(t-\tau) = h(\underset{\text{flip}}{-}(\tau \underset{\text{shift}}{-} t))$$

"flip and shift" view of convolution.



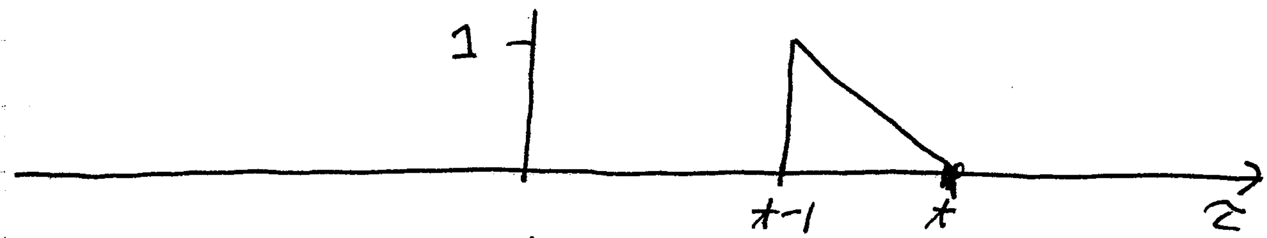


Example

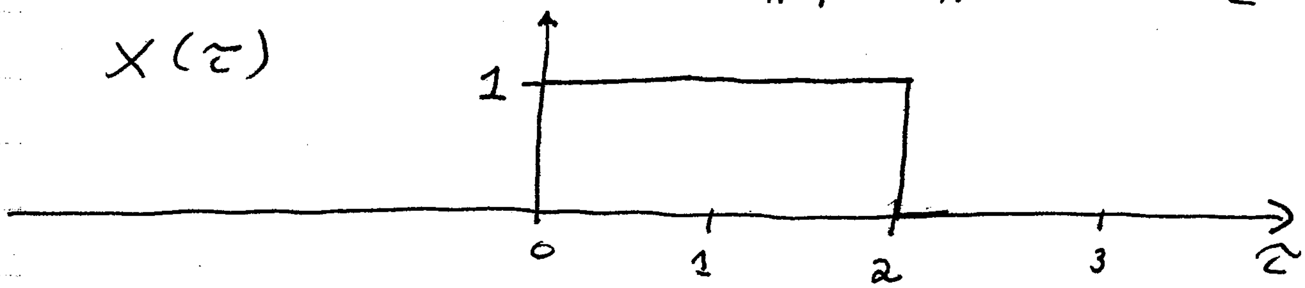
$$x(t) = u(t) - u(t-2)$$

$$h(t) = \big(u(t) - u(t-1)\big)\, t$$

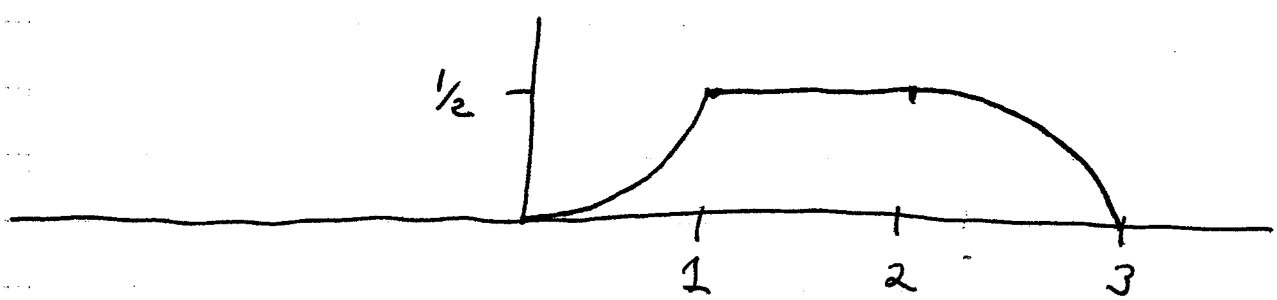
$h(t-\tau)$



$x(\tau)$



$$y(t) = \int x(\tau)\, h(t-\tau)\, d\tau$$



Memoryless LTI System

memoryless $\Leftrightarrow$ $h_n = 0$ for $n \neq 0$

$h(t) = 0$ for $t \neq 0$

$\Leftrightarrow$ $h_n = G\,\delta_n$

$h(t) = G\,\delta(t)$

for some gain factor G

Causal LTI Systems

$$\delta_n \rightarrow \boxed{T[\ ]} \rightarrow h_n$$

$$\delta(t) \rightarrow \boxed{T[\ ]} \rightarrow h(t)$$

causal $\Leftrightarrow$ $h_n = 0$ for $n < 0$

$h(t) = 0$ for $t < 0$

Stable LTI systems

An LTI system is BIBO stable

$$\Longleftrightarrow \quad \sum_{n=-\infty}^{\infty} |h_n| < \infty$$

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

proof

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$|y(t)| \le \left| \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right|$$

$$\le \int_{-\infty}^{\infty} |x(\tau)\, h(t-\tau)|\, d\tau$$

$$\le \int_{-\infty}^{\infty} M |h(t-\tau)|\, d\tau$$

$$\le M \int_{-\infty}^{\infty} |h(t-\tau)|\, d\tau$$

$$= M \int_{-\infty}^{\infty} |h(\tau)|\, d\tau = N$$

if $\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty \Rightarrow$ BIBO stability what about $\Leftarrow$? (proof by contradiction)

assume $\int_{-\infty}^{\infty} |h(t)|\, dt = \infty$

choose $x(t) = \text{sign}\{h(-t)\}$

$$= \begin{cases} 1 & h(-t) \geq 0 \\ -1 & h(-t) < 0 \end{cases}$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \text{sign}\{h(-\tau)\}\, h(t-\tau)\, d\tau$$

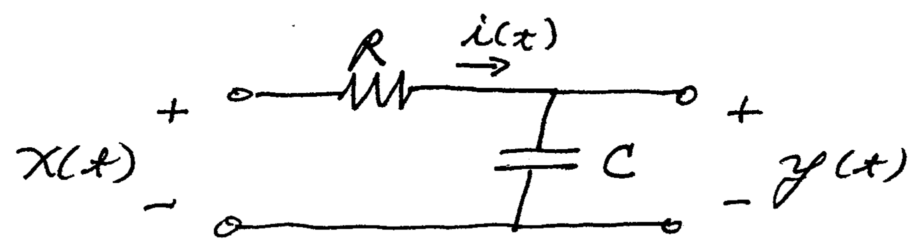
$$y(0) = \int_{-\infty}^{\infty} |h(-\tau)|\, d\tau = \infty$$

But since $|x(t)| < 2 \Rightarrow \cancel{BIBO}$

$$BIBO \Leftrightarrow \int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

LTI Analysis of Linear differential Equations

Consider the following system



We know

$$x(t) = R\,i(t) + y(t)$$

and

$$i(t) = C\frac{dy(t)}{dt}$$

So we have

$$x(t) = RC\frac{dy(t)}{dt} + y(t)$$

Define $\alpha \triangleq (RC)^{-1}$

$$\frac{dy(t)}{dt} + \alpha y(t) = \alpha x(t)$$

$$e^{\alpha t}\frac{dy(t)}{dt} + \alpha e^{\alpha t} y(t) = \alpha e^{\alpha t} x(t)$$

$$\frac{d}{dt}\left[e^{\alpha t} y(t) \right] = \alpha e^{\alpha t} x(t)$$

$$e^{\alpha t} y(t) = \int_{-\infty}^{t} \alpha e^{\alpha \tau} x(\tau)\, d\tau + K$$

K is a constant

$$y(t) = \int_{-\infty}^{t} \alpha\, e^{\alpha(\tau - t)} x(\tau)\, d\tau + K$$

$$= \int_{-\infty}^{t} \alpha\, e^{-\alpha(t-\tau)} x(\tau)\, d\tau + K$$

If the circuit ~~starts~~ starts at "rest", then

$$\lim_{t \to -\infty} y(t) = 0$$

$$\lim_{t \to -\infty} y(t) = \lim_{t \to -\infty} \int_{-\infty}^{t} \alpha e^{-\alpha(t-\tau)} x(\tau)\, d\tau + K$$

$$0 = 0 + K$$

$$\Rightarrow K = 0$$

So we have

$$y(t) = \int_{-\infty}^{t} \alpha e^{-\alpha(t-\tau)} x(\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{\infty} \alpha e^{-\alpha(t-\tau)} u(t-\tau)\, x(\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \underbrace{\alpha e^{-\alpha(t-\tau)} u(t-\tau)}_{h(t-\tau)}\, d\tau$$

Comments

1) This is an LTI system

2)

$$y(t) = x(t) * h(t)$$

impulse response

3)

$$h(t) = \alpha e^{-\alpha(t)} u(t)$$

$$= \alpha e^{-\frac{1}{RC}t} u(t)$$