Complex Exponential Inputs to LTI systems

$$x(t) \longrightarrow \boxed{T} \longrightarrow y(t)$$

$$y(t) = T[x(t)]$$

So we know that

$$y(t-d) = T[x(t-d)]$$

Let $\quad x(t) = e^{j\omega t}$

then

$$\begin{aligned} y(t-d) &= T[x(t-d)] \\ &= T[e^{j\omega(t-d)}] \\ &= T[e^{-j\omega d} e^{j\omega t}] \\ &= e^{-j\omega d}\, T[e^{j\omega t}] \end{aligned}$$

$$y(t-d) = e^{-j\omega d}\, y(t)$$

So we know that for $t=0$

$$y(-d) = e^{-j\omega d}\, y(0)$$

So setting $-d \to t$, we have

$$y(t) = e^{j\omega t}\, y(0)$$

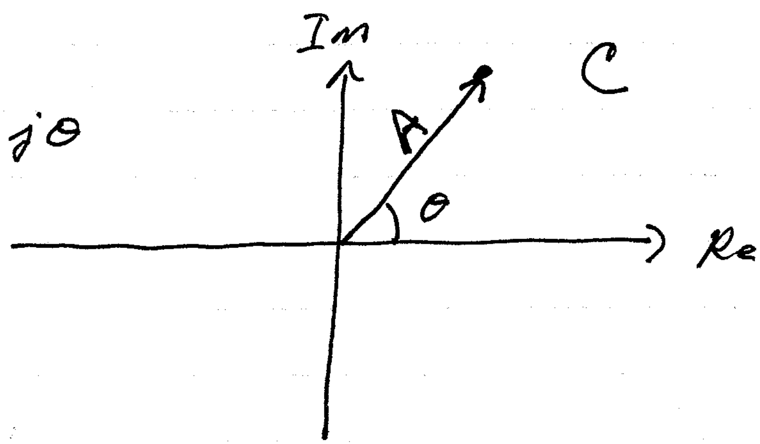
Conclusion:

$$\begin{pmatrix}\text{complex expodential} \\ \text{input}\end{pmatrix} \Longrightarrow \begin{pmatrix}\text{complex exponential} \\ \text{output}\end{pmatrix}$$

$$x(t) = e^{j\omega t} \longrightarrow \boxed{LTI} \longrightarrow y(t) = C e^{j\omega t}$$

where C is a complex number.

$C = Ae^{j\theta}$



Any complex number can be represented as

$$C = Ae^{j\theta}$$

Where A is the positive, real magnatude
θ is the angle in radians.

Sinusoidal Inputs to LTI Systems

$$x(t) = \cos(\omega t) \longrightarrow \boxed{T[\cdot]} \longrightarrow y(t) = ?$$

$$x(t) = \cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\begin{aligned}
y(t) &= T[x(t)] \\
&= T\left[\tfrac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)\right] \\
&= \tfrac{1}{2}\, T[e^{j\omega t}] + \tfrac{1}{2}\, T[e^{-j\omega t}] \\
&= \tfrac{1}{2}\, C_1 e^{j\omega t} + \tfrac{1}{2}\, C_2 e^{-j\omega t}
\end{aligned}$$

If $y(t)$ is real

$$\operatorname{Im}\left\{ \frac{C_1 e^{j\omega t} + C_2 e^{-j\omega t}}{2} \right\} = 0$$

$$C_1 = A_1 e^{j\theta_1}$$
$$C_2 = A_2 e^{j\theta_2}$$

$$\operatorname{Im}\left\{ A_1 e^{j(\omega t+\theta_1)} + A_2 e^{j(\theta_2 - \omega t)} \right\} = 0$$

$\forall t$ $$A_1 \sin(\omega t + \theta_1) + A_2 \sin(\theta_2 - \omega t) = 0$$

$$A_1 \sin(\omega t + \theta_1) - A_2 \sin(\omega t - \theta_2) = 0$$

$$\Longleftrightarrow \quad A_1 = A_2 \quad \theta_1 = -\theta_2$$

$$\Longleftrightarrow \quad C_1 = C_2^*$$

$$y(t) = \frac{1}{2}\left\{ \frac{c e^{j\omega t} + c^* e^{j\omega t}}{\cancel{2}} \right\}$$

$$\left(c = A e^{j\theta} \right) = \frac{1}{2}\left\{ A e^{j(\omega t+\theta)} + A e^{-j(\omega t+\theta)} \right\}$$

$$= A \cos(\omega t + \theta)$$

Conclusion: For and LTI system, and input of $\cos(\omega t)$ gives an output of

$$A \cos(\omega t + \phi)$$

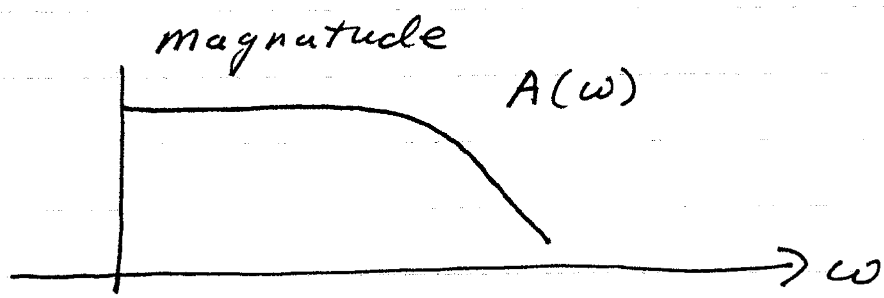
So we have that

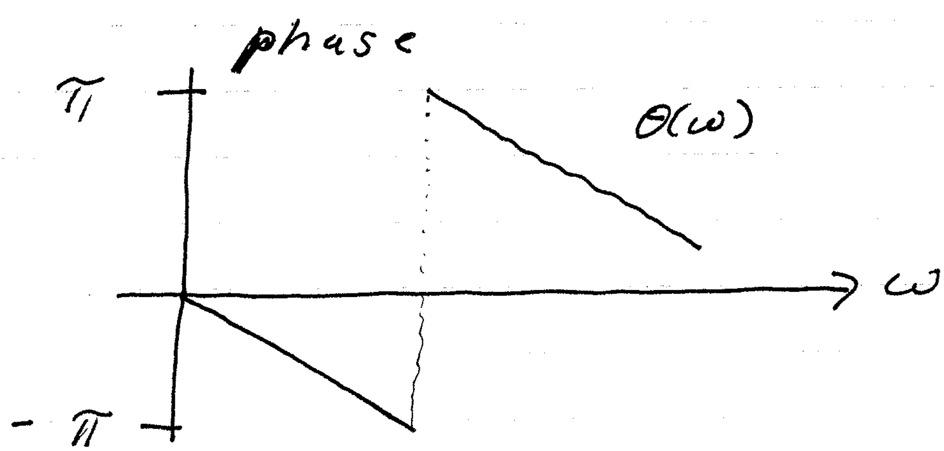
$$x(t) = \cos(\omega t) \longrightarrow \boxed{T[\cdot]} \longrightarrow y(t) = A\cos(\omega t + \theta)$$

where A and θ are constants which depend on the frequency ω.

Idea

Plot A and θ as a function of frequency ω.





Calculating Transfer Functions of LTI Systems

$$x(t)=e^{j\omega t} \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h(t)\end{matrix}} \longrightarrow y(t)=?$$

$$y(t) = C(\omega)\, e^{j\omega t}$$

What is $C(\omega)$?

We know that

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega(t-\tau)}\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega t}\, e^{-j\omega\tau}\, d\tau$$

$$= e^{j\omega t} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau}_{C(\omega)}$$

$$C(\omega) = \underbrace{\int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt}_{\text{Fourier Transform}}$$

Also notice that if $h(t)$ is <u>real</u>, then

$$C(-\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j(-\omega)t}\, dt$$

$$= \int_{-\infty}^{\infty} h(t) \left(e^{-j\omega t}\right)^{*} dt$$

$$= \left(\int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt\right)^{*}$$

$$= C^{*}(\omega)$$

$$C(-\omega) = C^{*}(\omega)$$

Summary

For an LTI system $y(t) = S[x(t)]$

1) When $x(t) = e^{j\omega t}$, then

$$y(t) = C(\omega)\, e^{j\omega t}$$

where $C(\omega)$ depend on frequency ω.

2) $$C(\omega) = \underbrace{\int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt}_{\text{Fourier Transform}}$$

3) If the system is real, then

$$C(-\omega) = C^*(\omega)$$

4) If the system is real and $x(t) = \cos(\omega t)$ then

$$y(t) = |C(\omega)| \cos(\omega t + \angle C(\omega))$$