Signals

Represented by a function of one or more independent variables

$f(t) \quad 0 \le t \le \infty$

independent variable

$x(k) \quad k = 0, 1, \ldots \infty$ also written x_k

Continuous time signal - Continuously valued independent variable

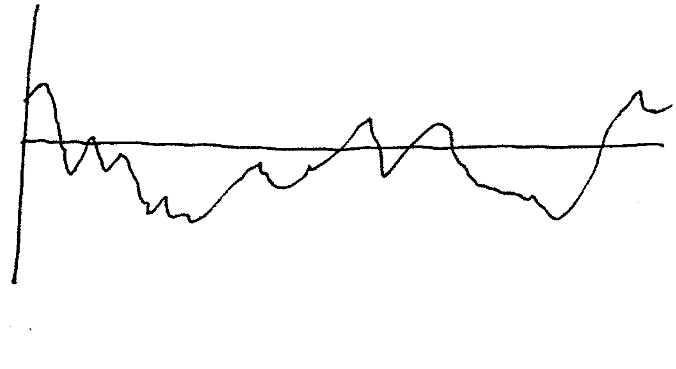
Discrete time signal - Discrete (integer) valued independent variable

Digital signal - Discrete time signal with discrete values.

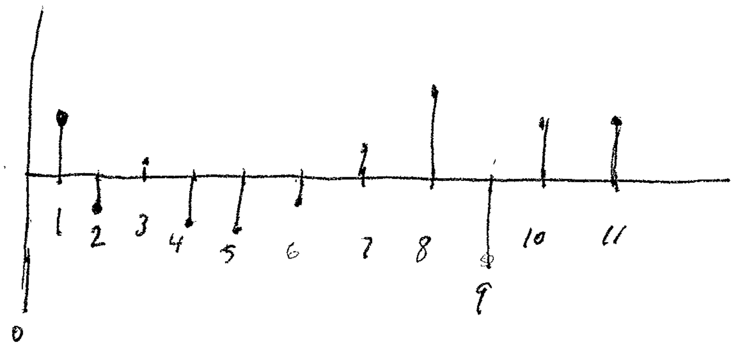
warning: continuous time $\not\Leftrightarrow$ continuous function

Examples:

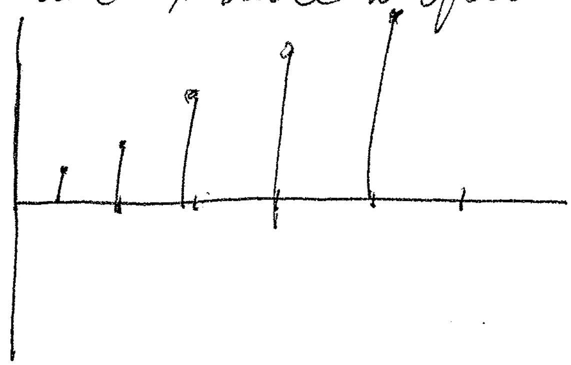
1. Audio Signal (e.g. speech, music) - continuous time, single indep. variable



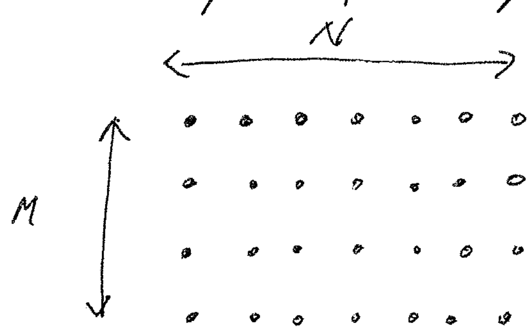
2. Digital audio Recording (e.g. CD, DAT)
- discrete time, single indep variable, discrete value



3. Annual Trade Deficit - discrete time



4. Computer Displayed Image -
discrete space, two parameter

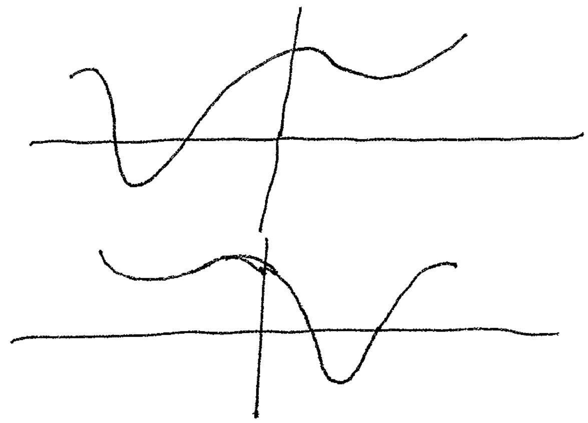


a point of the image is called a pixel - picture element

Signal Transformations

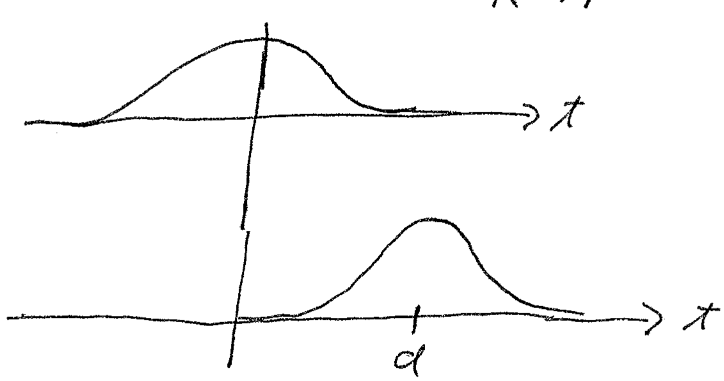
Time reversal

$$x(t) \longrightarrow x(-t)$$
$$x_k \longrightarrow x_{-k}$$



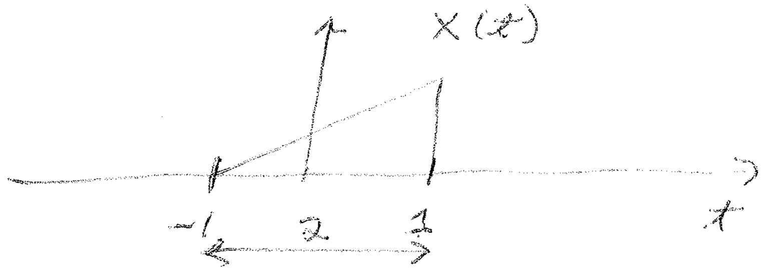
Time delay

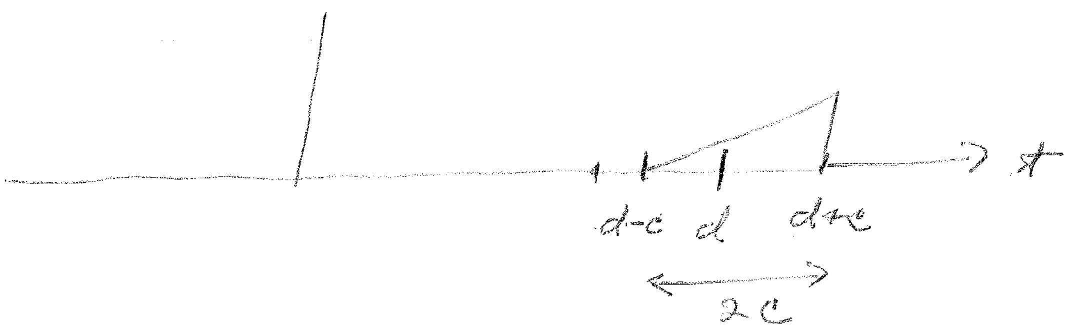
$$x(t) \longrightarrow x(t-d)$$
$$x_k \longrightarrow x_{k-m}$$



Time Scaling

$$x(t) \longrightarrow x\left(\frac{t-d}{c}\right)$$

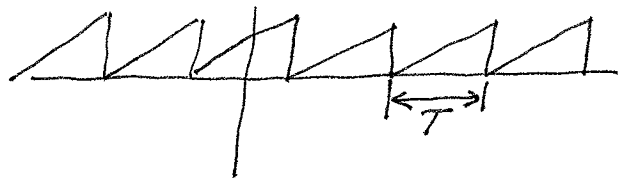




periodic signals

$x(t)$ is periodic with period T if

$$x(t) = x(t+T)$$



Since $x(t) = x(t+T)$ $\quad u = t+T$
$= x(u)$
$= x(u+T)$
$= x(t+2T)$

$\Rightarrow x(t) = x(t+kT)$ k an integer

Therefor $x(t)$ is periodic with period kT for all positive integers k.

Definition: The fundamental period of $x(t)$ is the smallest positive value of T.

Discrete case

$x_k = x_{k+N} \Rightarrow$ periodic with period N

Fundamental period is the smallest value of N.

Even/Odd Signals

Even signal

$$X(t) = X(-t) \quad \text{Continuous time}$$

$$X(n) = X(-n) \quad \text{Discrete time}$$

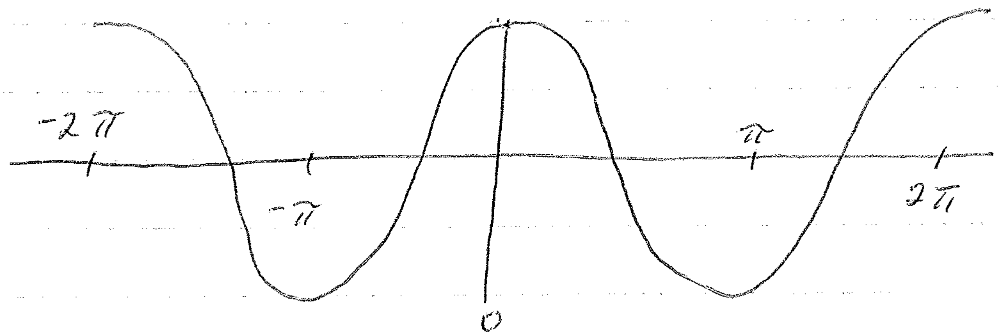
Odd Signal

$$X(t) = -X(-t) \quad \text{Continuous time}$$

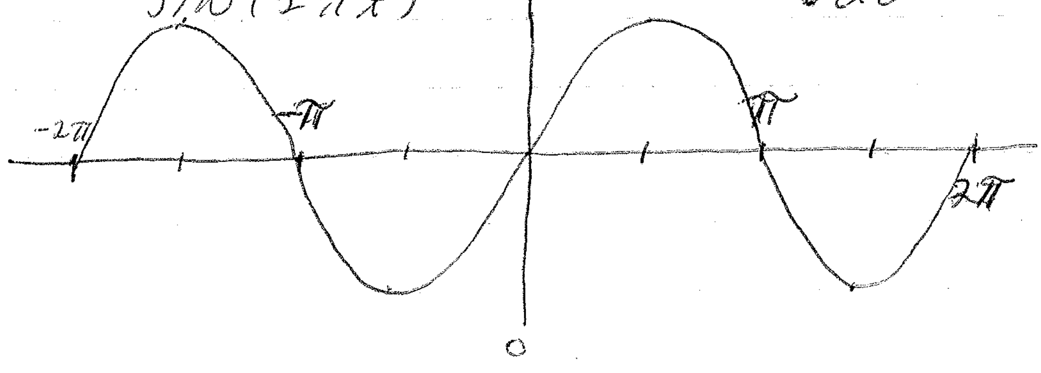
$$X(n) = -X(-n) \quad \text{Discrete Time}$$

examples:

$\cos(2\pi t)$ even



$\sin(2\pi t)$ odd



Energy/Power

Energy

$$E = \int_{-\infty}^{\infty} (x(t))^2 \, dt \qquad CT$$

$$E = \sum_{n=\infty}^{\infty} (x(n))^2 \qquad DT$$

Power

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (x(t))^2 \, dt \qquad CT$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (x(n))^2 \qquad DT$$

Average

$$Ave = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt \quad CT$$

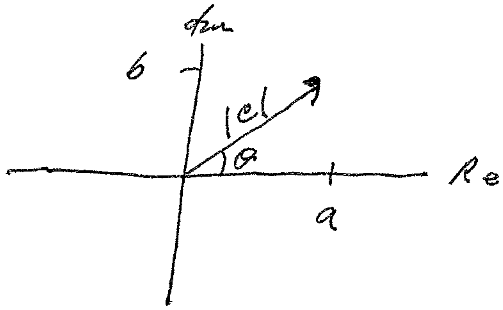
$$Ave = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)$$

Exponential Signals:

continuous time:

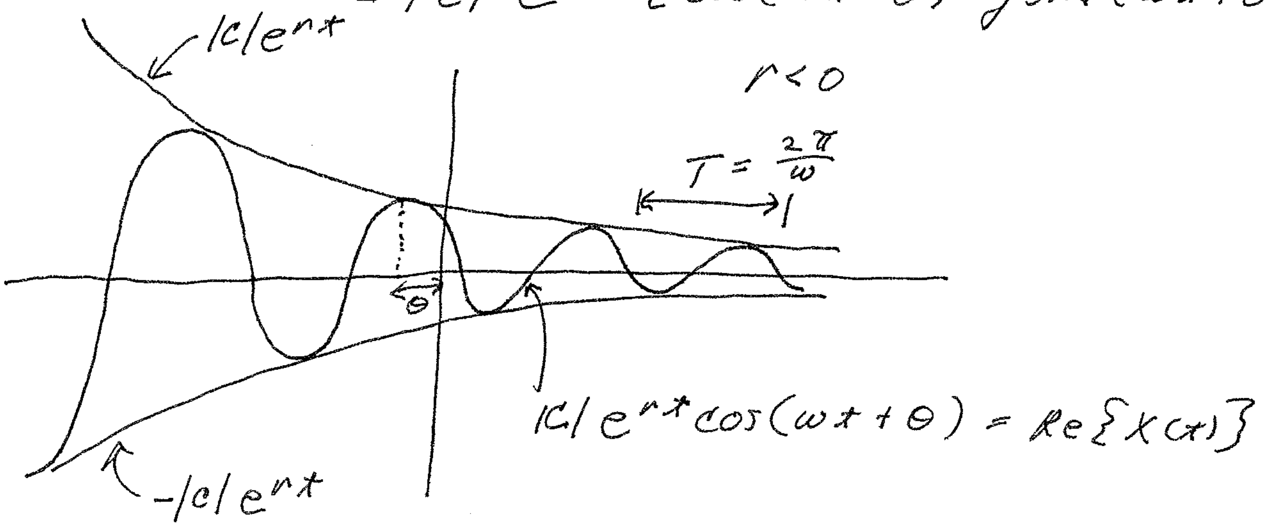
example 1) $x(t) = c\, e^{st}$ $\qquad s = r + j\omega$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad c = a + ib$



$$
\begin{aligned}
c &= a + ib \\
&= |c|\cos\theta + j|c|\sin\theta \\
&= |c|\, e^{j\theta}
\end{aligned}
$$

$$
\begin{aligned}
x(t) &= |c|\, e^{j\theta} e^{st} \\
&= |c|\, e^{rt} e^{j(\omega t + \theta)} \\
&= |c|\, e^{rt} \{\cos(\omega t + \theta) + j\sin(\omega t + \theta)\}
\end{aligned}
$$



$$\cos(\omega(t+T) + \theta) = \cos(\omega t + \theta)$$

$$
\begin{aligned}
\Rightarrow \quad \omega T &= 2\pi \\
T &= \frac{2\pi}{\omega}
\end{aligned}
$$

example 2) $\quad x_k = c\alpha^k$

$$c = a + jb = |c| e^{j\theta}$$
$$\alpha = e^{s} = e^{r} e^{j\omega}$$

$$x_k = |c| e^{j\theta} (e^{r} e^{j\omega})^k$$

$$= |c| e^{rk} e^{j(\omega k + \theta)}$$

<u>Discrete Time periodic exponentials</u>

$$x_k = e^{j\omega k}$$

Is x_k periodic?

$$e^{j\omega(k+N)} \overset{?}{=} e^{j\omega k}$$

$\Longleftrightarrow \quad \omega N = m 2\pi$ for m an integer

$$\boxed{\text{periodic} \Longleftrightarrow \quad \omega = \frac{m}{N} 2\pi \quad \text{where } N \text{ is the period}}$$

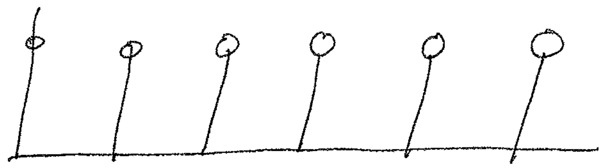
Let $\hat{\omega} = \omega + 2\pi$

$$e^{j\hat{\omega}k} = e^{j\omega k + j2\pi k}$$
$$= e^{j\omega k} e^{j2\pi k}$$
$$= e^{j\omega k}$$

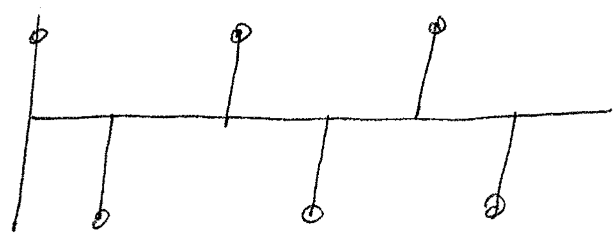
$$\boxed{\text{Frequencies of } \omega \text{ and } \omega + 2\pi \text{ are equivalent}}$$

examples:

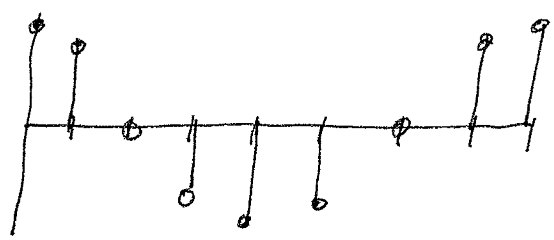
$$\left.\begin{array}{c}\omega = 0 \\ \text{or} \\ \omega = 2\pi\end{array}\right\} \Rightarrow e^{j\omega k} = 1$$



$$\omega = \pi \Rightarrow e^{j\omega k} = (-1)^k$$

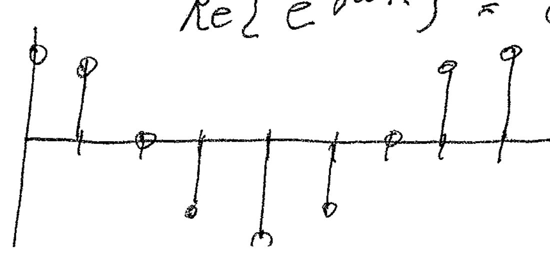


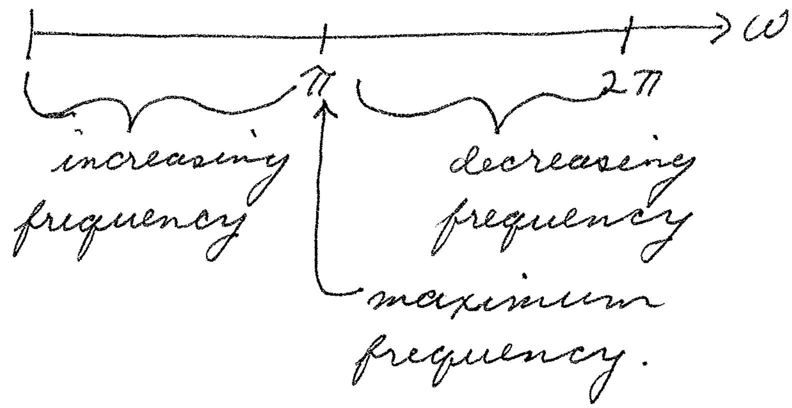
$$\omega = \tfrac{1}{8} 2\pi \qquad Re\{e^{j\omega k}\} = \cos(\omega k)$$



$$\omega = \tfrac{7}{8} 2\pi \Leftrightarrow \hat{\omega} = \tfrac{7}{8} 2\pi - 2\pi$$
$$= -\tfrac{1}{8} 2\pi$$

$$e^{j\omega k} = e^{j\omega k} = e^{-\frac{1}{8} 2\pi k}$$
$$Re\{e^{j\omega k}\} = \cos(\tfrac{1}{8} 2\pi k)$$

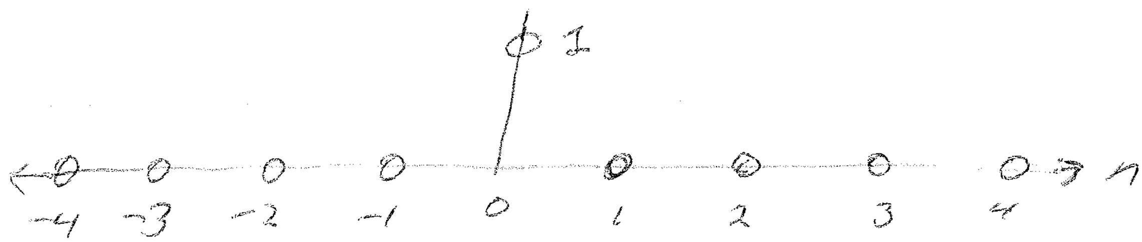



ω
π
2π
increasing frequency
decreasing frequency
maximum frequency.

Impulse and Step Functions

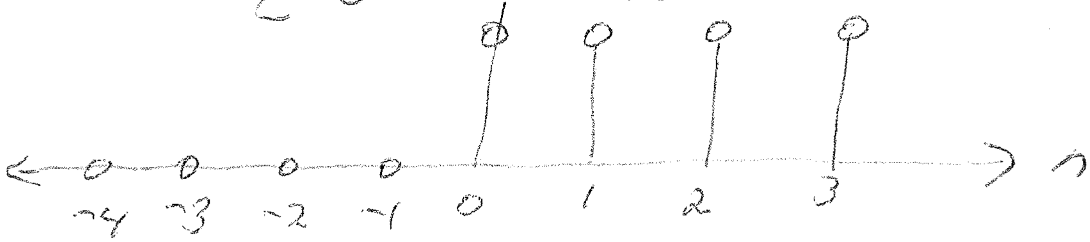
- Discrete time Impulse

$$\delta_n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$



- Discrete time step functions

$$u_n = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



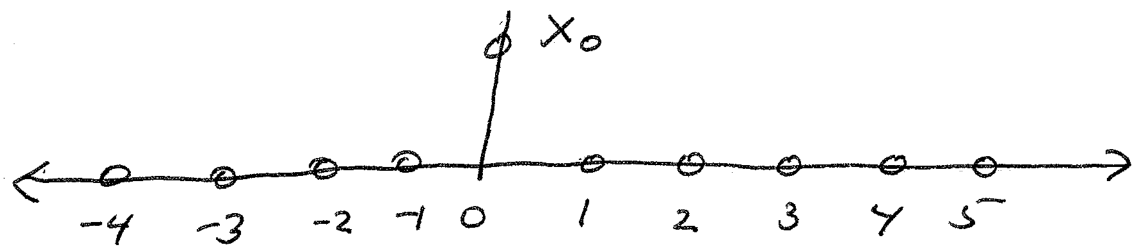
- Relationship between δ_n and u_n

$$\delta_n = u_n - u_{n-1}$$

n	$u_n - u_{n-1}$
-1	$0 - 0 = 0$
0	$1 - 0 = \underline{1}$
1	$1 - 1 = 0$
2	$1 - 1 = 0$

- Multiplication by DT impulses

$$x_n \delta_n = \begin{cases} x_0 & n=0 \\ 0 & \text{otherwise} \end{cases}$$



$$\sum_{n=-\infty}^{\infty} x_n \delta_n = x_0$$

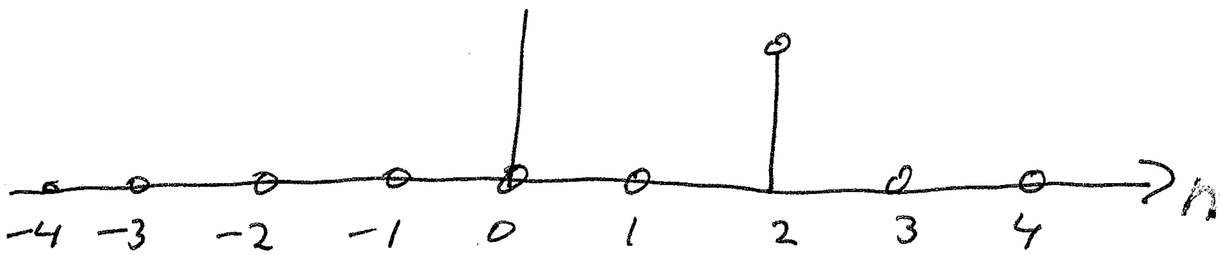
property 1

$$\sum_{n=-\infty}^{\infty} x_n \delta_n = x_0$$

∘ Representation of DT signals with DT impulses

$$\delta_{n-k} = \begin{cases} 1 & \text{for } n=k \\ 0 & \text{otherwise} \end{cases}$$

δ_{n-2}



So $$X_n \delta_{n-k} = \begin{cases} X_k & \text{for } n=k \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$\sum_{n=-\infty}^{\infty} X_n \delta_{n-k} = \cdots + 0 + 0 + X_k + 0 + 0 + \cdots$$

$$= X_k$$

So we know that

$$\sum_{n=-\infty}^{\infty} x_n \delta_{n-k} = x_k$$

Since δ_n is a symmetric function, we know that $\delta_{n-k} = \delta_{k-n}$.
Therefore,

$$\sum_{n=-\infty}^{\infty} x_n \delta_{k-n} = x_k$$

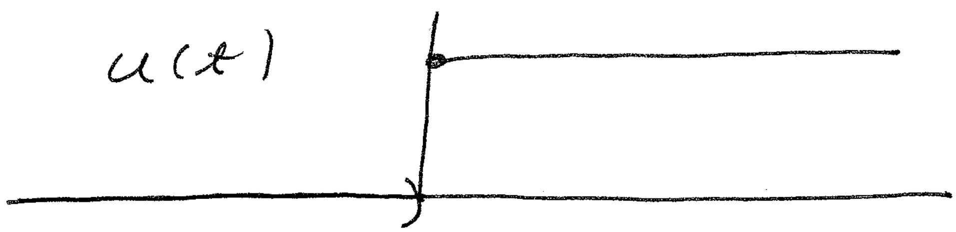
Substituting variables $\left\{ \begin{matrix} k \to n \\ n \to k \end{matrix} \right\}$,
we have

$$\sum_{k=-\infty}^{\infty} x_k \delta_{n-k} = x_n$$

$$\boxed{\text{property 2} \qquad \sum_{k=-\infty}^{\infty} x_k \delta_{n-k} = x_n}$$

• Continuous time step function

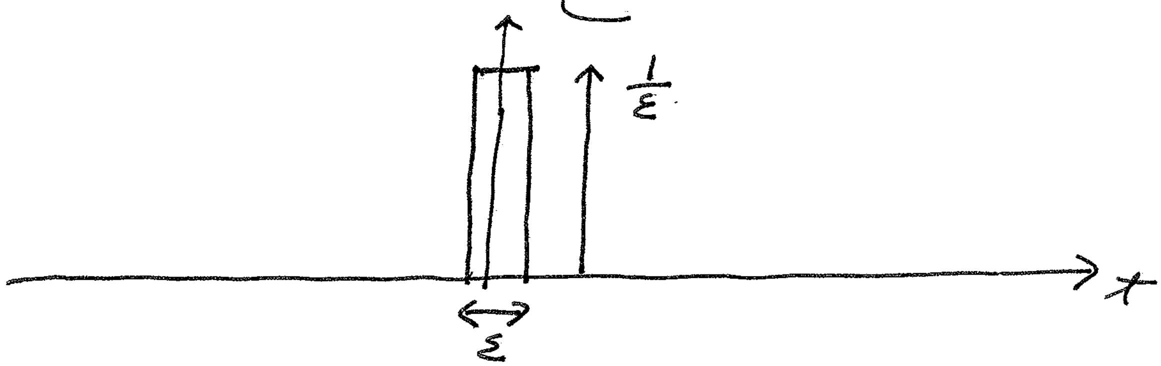
$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



- Continuous time Impulse

  - Consider the function

$$\delta_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon} & \text{for } |t| < \varepsilon/2 \\ 0 & \text{otherwise} \end{cases}$$



Comments

- As $\varepsilon \to 0$ spike becomes large

- ... But area remains constant

$$\int_{-\infty}^{\infty} \delta_\varepsilon(t)\, dt = \int_{-\varepsilon/2}^{\varepsilon/2} \frac{1}{\varepsilon}\, dt = 1$$

Consider

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \delta_\varepsilon(t)\, X(t)\, dt$$

$$= \lim_{\varepsilon \to 0} \int_{-\varepsilon/2}^{\varepsilon/2} X(t) \frac{1}{\varepsilon}\, dt = X(0)$$

(assumes that $X(t)$ is continuous)

Definition: Delta function (Not rigorous!)

$$\delta(t) \triangleq \lim_{\varepsilon \to 0} \delta_\varepsilon(t)$$

property 1

$$\boxed{\int_{-\infty}^{\infty} x(t)\, \delta(t)\, dt = x(0)}$$

In order to derive property 2, let

$$x(t) \triangleq y(d-t)$$

then we have (by prop 1)

$$\int_{-\infty}^{\infty} y(d-t)\, \delta(t)\, dt = y(d-0)$$

Make the change of variables

$$\tau \triangleq d-t \Rightarrow t = d-\tau$$

Then

$$\int_{-\infty}^{\infty} y(\tau)\, \delta(d-\tau)\, d\tau = y(d)$$

Substituting t for d yields

$$\boxed{\begin{array}{l}\text{property 2}\\ \displaystyle\int_{-\infty}^{\infty} y(\tau)\,\delta(t-\tau)\,d\tau = y(t)\end{array}}$$

This is the decomposition of signal into impulses.