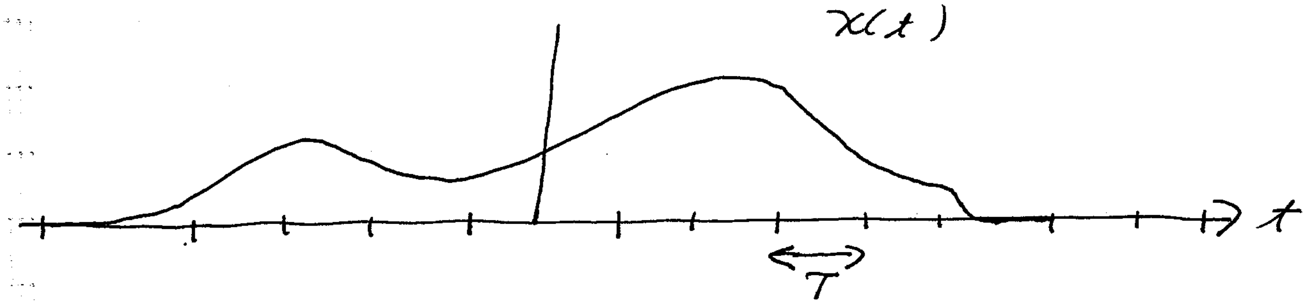
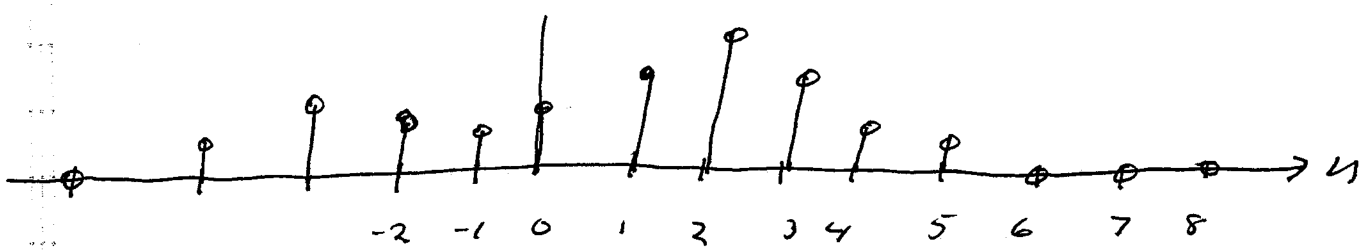
Sampling and Reconstruction

$$x(t) \longrightarrow \boxed{\text{sample at period } T} \longrightarrow y[n] = x(nT)$$





How do $X(\omega)$ and $Y(\omega)$ relate?

$$x(t) \overset{CTFT}{\Longleftrightarrow} X(\omega)$$

$$y[n] \overset{DTFT}{\Longleftrightarrow} Y(\omega)$$

$$y[n] = x(nT)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{+jnT\omega}\, d\omega$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} X\left(\omega + k\frac{2\pi}{T}\right) e^{+jnT\left(\omega + k\frac{2\pi}{T}\right)}\, d\omega$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} X\left(\omega + k\frac{2\pi}{T}\right) e^{+jnT\omega}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \sum_{k=-\infty}^{\infty} X\left(\omega + k\frac{2\pi}{T}\right) e^{+jnT\omega}\, d\omega$$

$$\Omega = T\omega \qquad \omega = \frac{\Omega}{T}$$
$$d\Omega = T\,d\omega \qquad d\omega = \frac{d\Omega}{T}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{\left[\frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\Omega + k2\pi}{T}\right)\right]}_{Y(\omega)} e^{+jn\Omega}\, d\Omega$$

$$y[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(\omega)\, e^{+jn\omega}\, d\omega$$

Sampling relationship

$$y[n] = x(nT)$$

$$Y(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\underbrace{\frac{\omega + k2\pi}{T}}_{\text{CT Frequency}}\right)$$

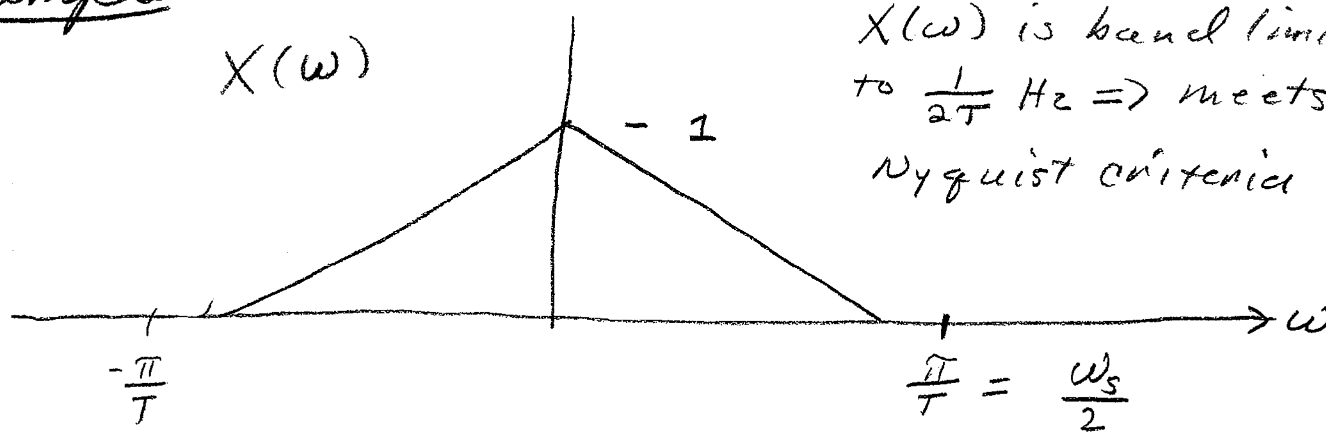
Conclusion:

$$y[n] = x(nT)$$

$$\Updownarrow$$

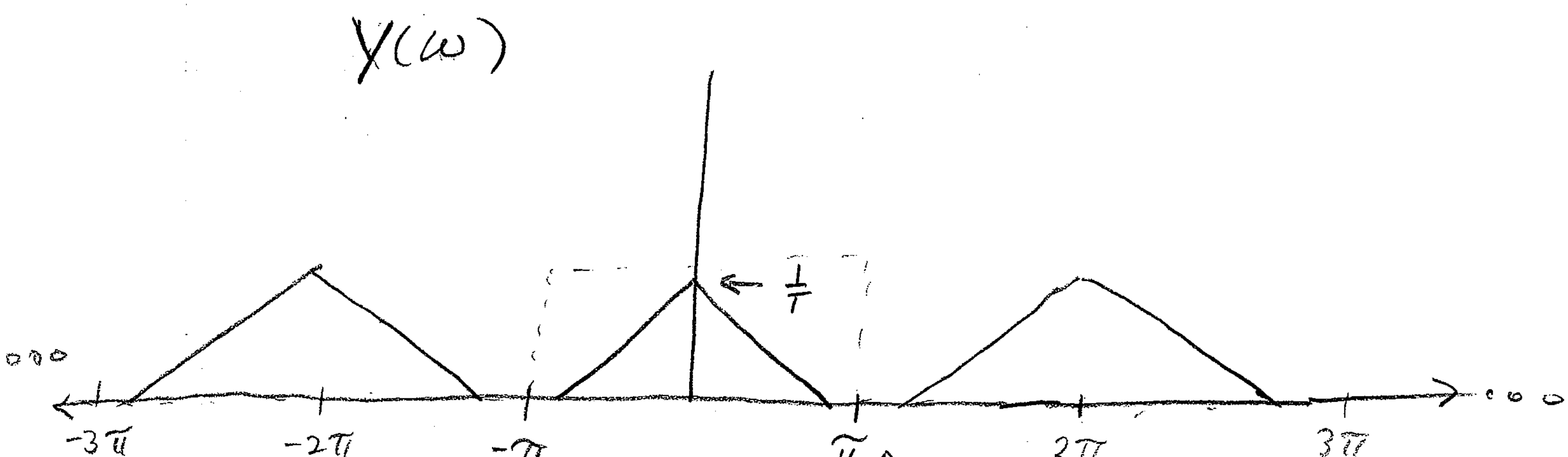
$$Y(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega + k2\pi}{T}\right)$$

Example



$X(\omega)$ is band limited to $\frac{1}{2T}$ Hz $\Rightarrow$ meets Nyquist criteria

where ω_s - sampling rate in rad/sec



No aliasing because signal is bandlimited to $\frac{1}{2T}$

Reconstruction

$$y[n] \longrightarrow \boxed{\begin{array}{c}\text{impulse}\\ \text{generator}\\ T\end{array}} \longrightarrow s(t)$$

$$s(t) = \sum_{k=-\infty}^{\infty} y[k]\, \delta(t - kT)$$

What does $S(\omega) = ?$

$$S(\omega) = \int_{-\infty}^{\infty} s(t)\, e^{-j\omega t}\, dt \qquad (\text{CTFT})$$

$$= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} y[k]\, \delta(t-kT)\, e^{-j\omega t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} y[k] \int_{-\infty}^{\infty} \delta(t-kT)\, e^{-j\omega t}\, dt$$

$$= \underbrace{\sum_{k=-\infty}^{\infty} y[k]\, e^{-j\omega kT}}_{\text{DTFT}}$$

$$= Y(\omega T)$$

Reconstruction Relationship

$$s(t) = \sum_{k=-\infty}^{\infty} y[k]\,\delta(t-kT)$$

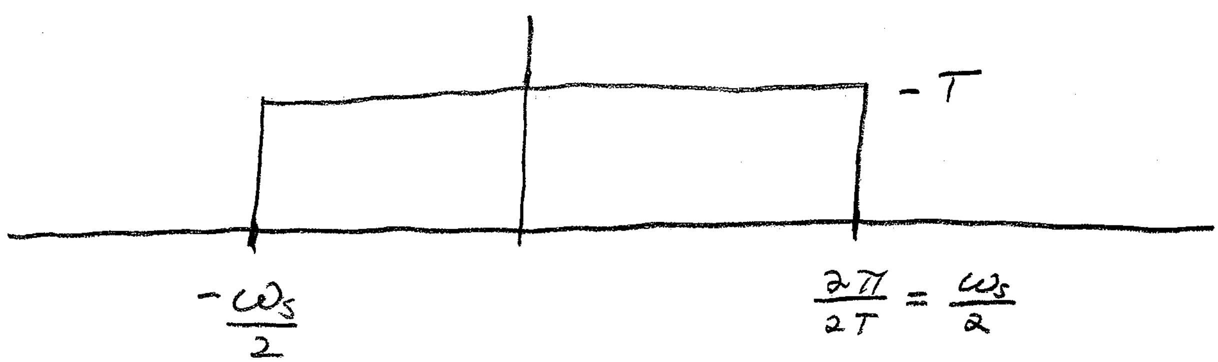
$$S(\omega) = Y(\omega T)$$

Reconstruction Filter

$$s(t) \longrightarrow \boxed{H(\omega)} \longrightarrow z(t)$$

$$H(\omega) = \begin{cases} T & \text{for } |\omega| < \frac{2\pi}{2T} \\ 0 & \text{for } |\omega| > \frac{2\pi}{2T} \end{cases}$$

$h(t) = \mathrm{sinc}(t/T)$



Example

$$x(t) = \text{sinc}(t) \qquad T = 1/2$$

$$x(t) \longrightarrow \boxed{\begin{matrix}\text{Sample}\\ T\end{matrix}} \longrightarrow y[n]$$

(CTFT) $\quad X(\omega) = \text{rect}(\omega/(2\pi))$

(DTFT) $\quad Y(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{T}\right)$

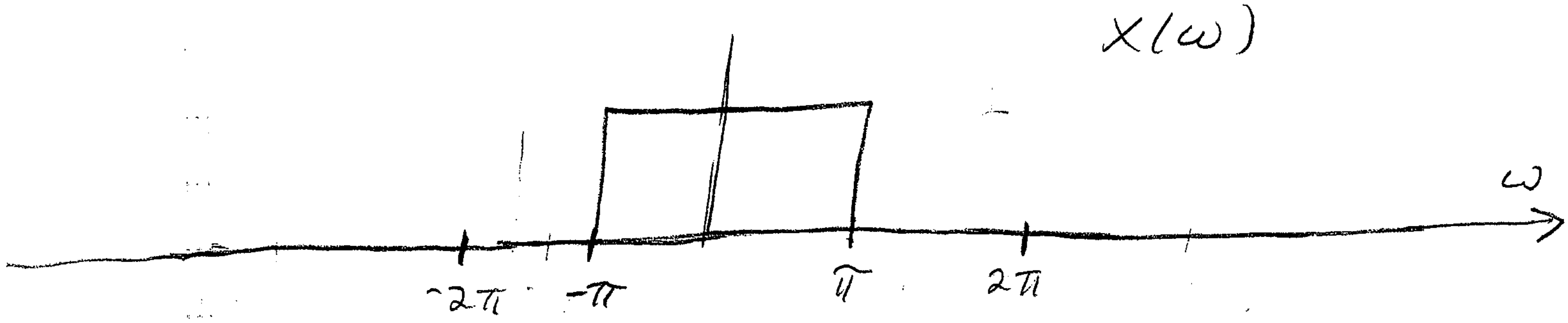
$$= \frac{1}{(1/2)} \sum_{k=-\infty}^{\infty} \text{rect}\left(\frac{\omega - 2\pi k}{2\pi(1/2)}\right)$$

$$= 2 \sum_{k=-\infty}^{\infty} \text{rect}\left(\frac{\omega - 2\pi k}{\pi}\right)$$

(CTFT) $\quad S(\omega) = Y(\omega T) = Y(\omega/2)$

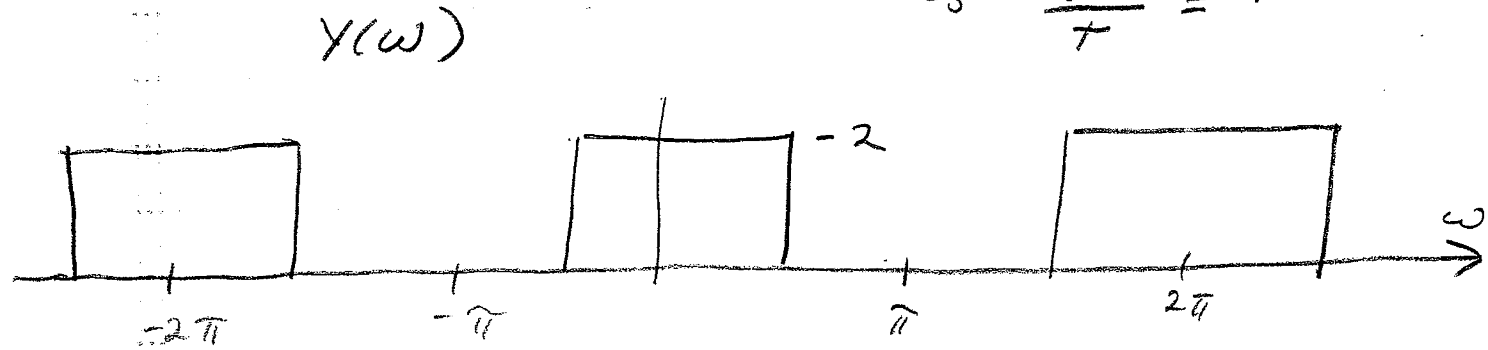
$$= 2 \sum_{k=-\infty}^{\infty} \text{rect}\left(\frac{\omega/2 - 2\pi k}{\pi}\right)$$

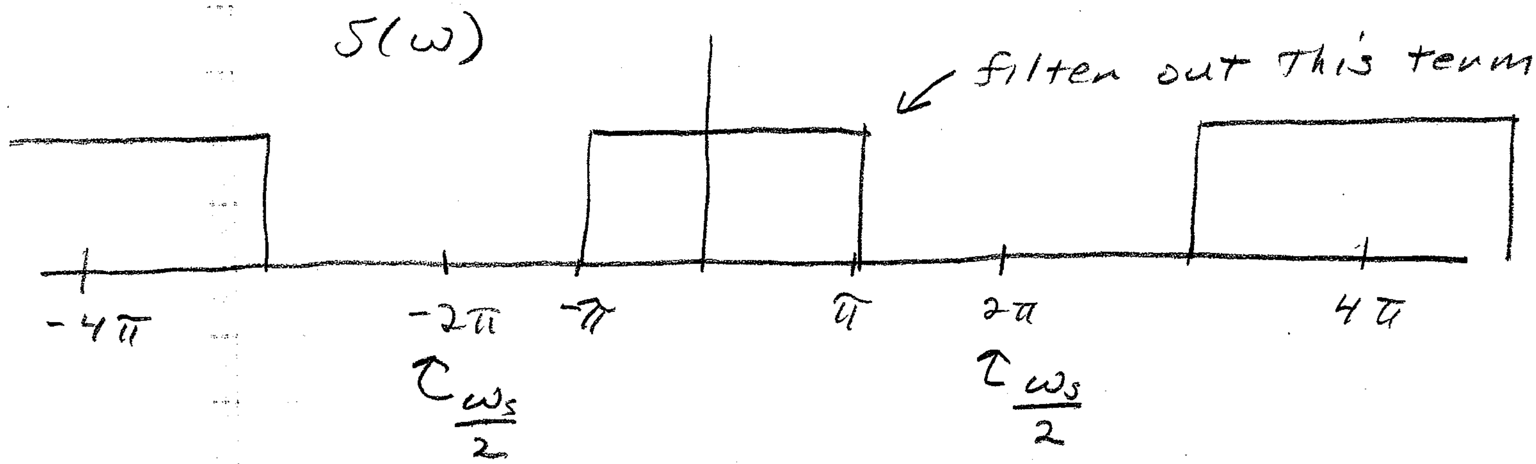
$$= 2 \sum_{k=-\infty}^{\infty} \text{rect}\left(\frac{\omega - 4\pi k}{2\pi}\right)$$

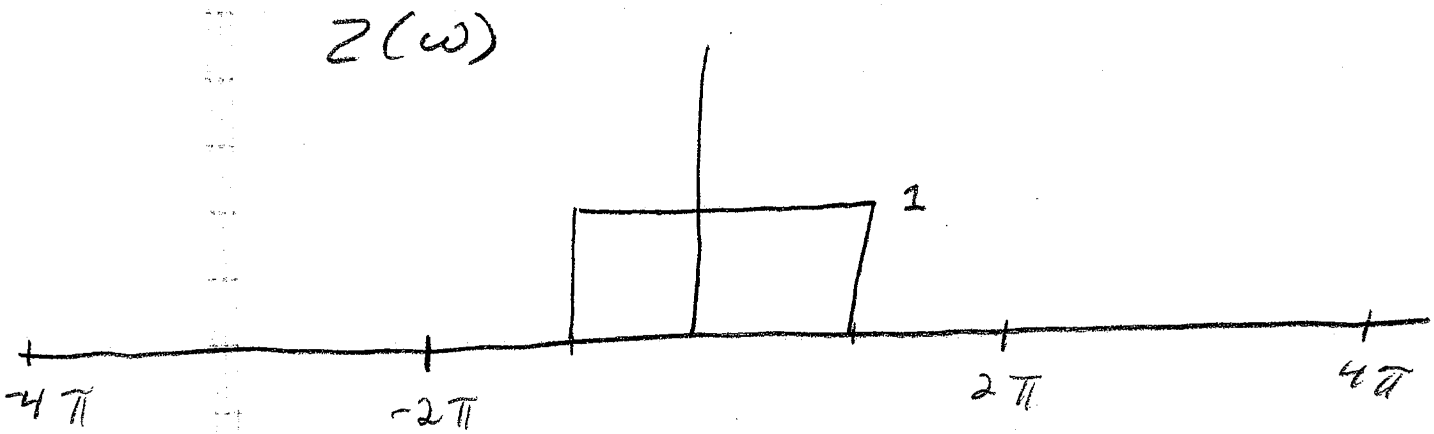


sampling frequency $= f_s = \frac{1}{T} = 2$

$\omega_s = \frac{2\pi}{T} = 4\pi$







$\boxed{\omega_s > 2\omega_M}$

$x(t),\ X(\omega)$

$\downarrow$ Sample with rate T

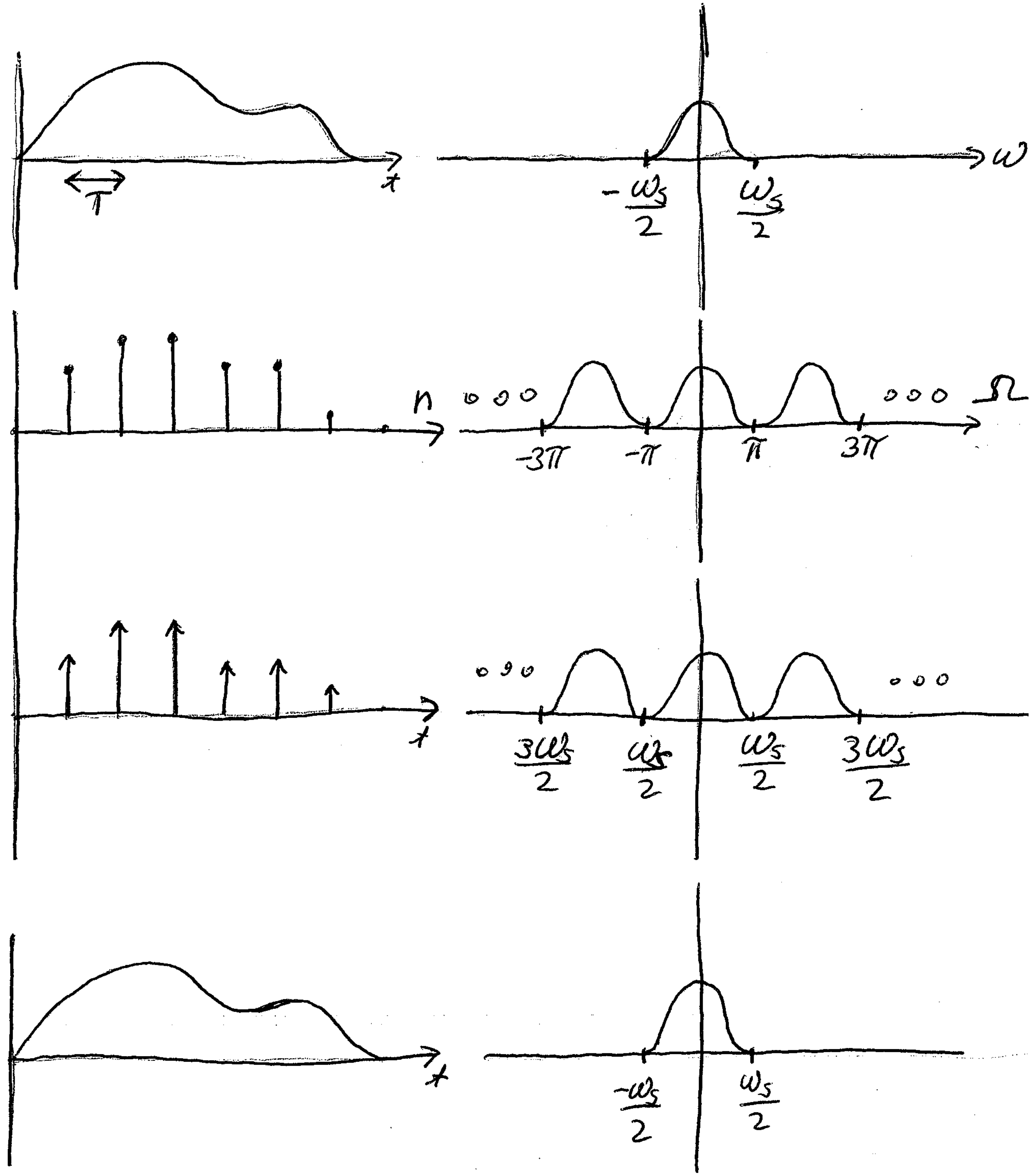
$\tilde{x}(n),\ \tilde{X}(\Omega)$

$\downarrow$ Generate Pulses $p(t) = T\delta(t)$

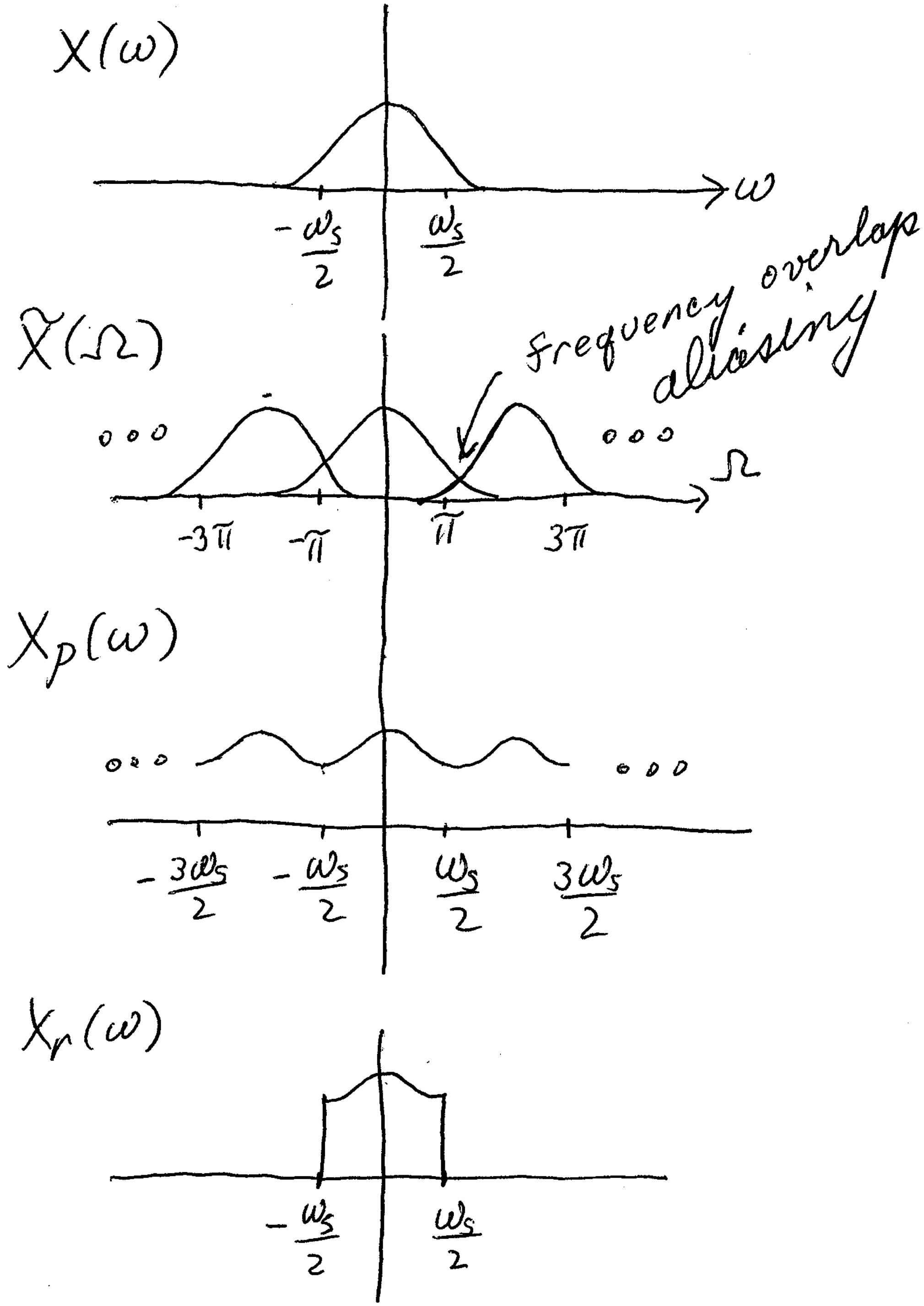
$x_p(t),\ X_p(\omega)$

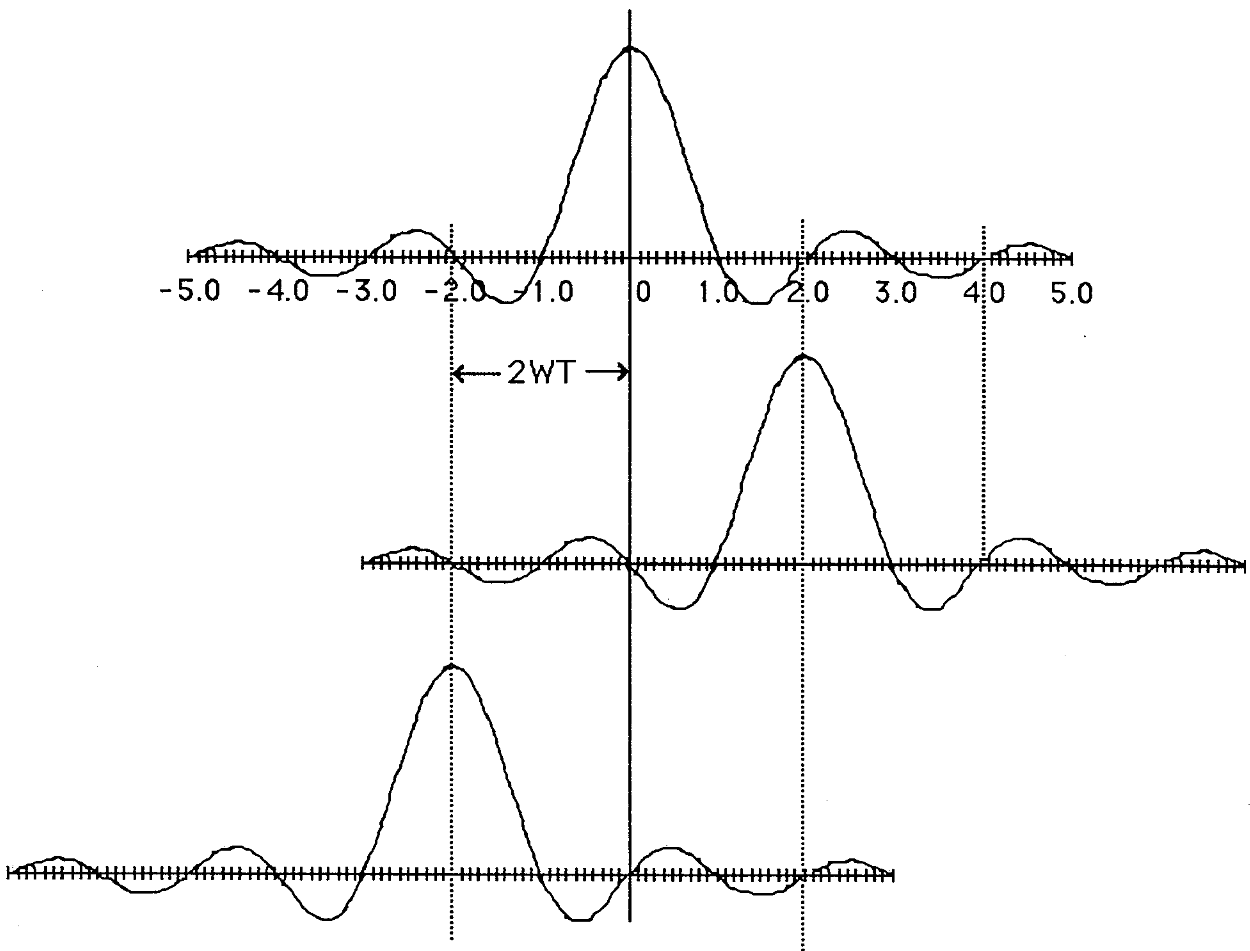
$\downarrow$ Filter $H(\omega) = \mathrm{rect}(\omega T/\pi)$, $h(t) = \frac{1}{T}\mathrm{sinc}(t/T)$

$x_r(t),\ X_r(\omega)$

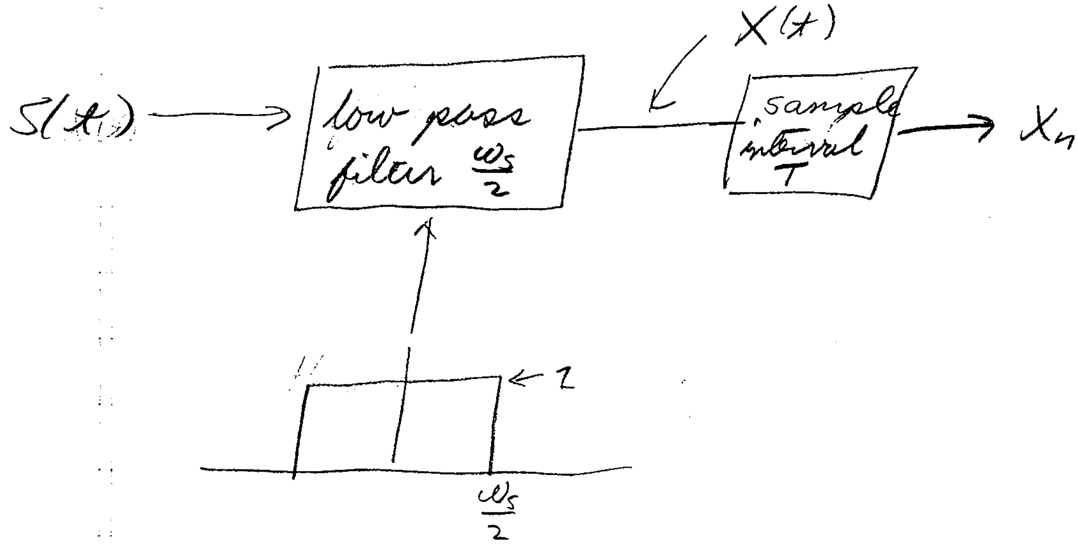


$\omega_s < 2\omega_M$

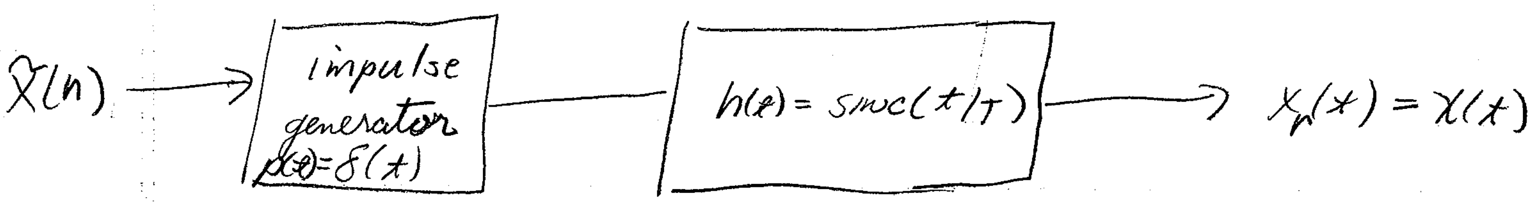



-5.0 -4.0 -3.0 -2.0 -1.0 0 1.0 2.0 3.0 4.0 5.0
2WT

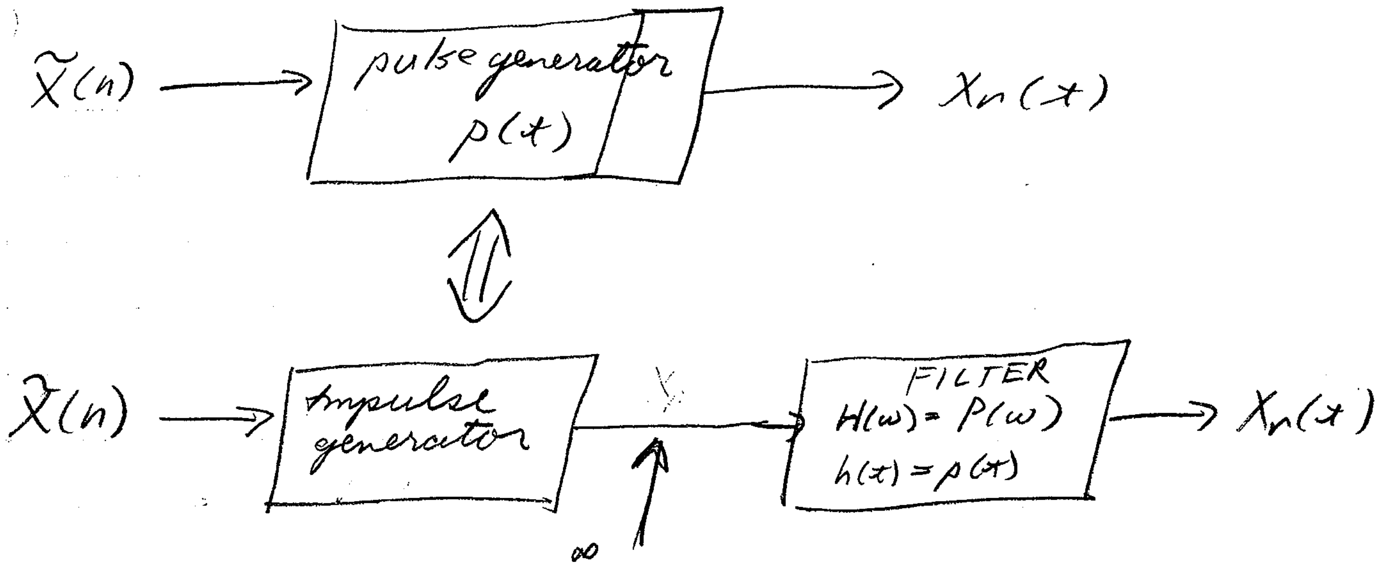
Sampling system



Reconstruction



More general reconstruction



$$X_\delta(t) = \sum_{n=-\infty}^{\infty} \tilde{X}(n)\,\delta(t-nT)$$

$$X_r(\omega) = X_\delta(\omega)\, P(\omega)$$

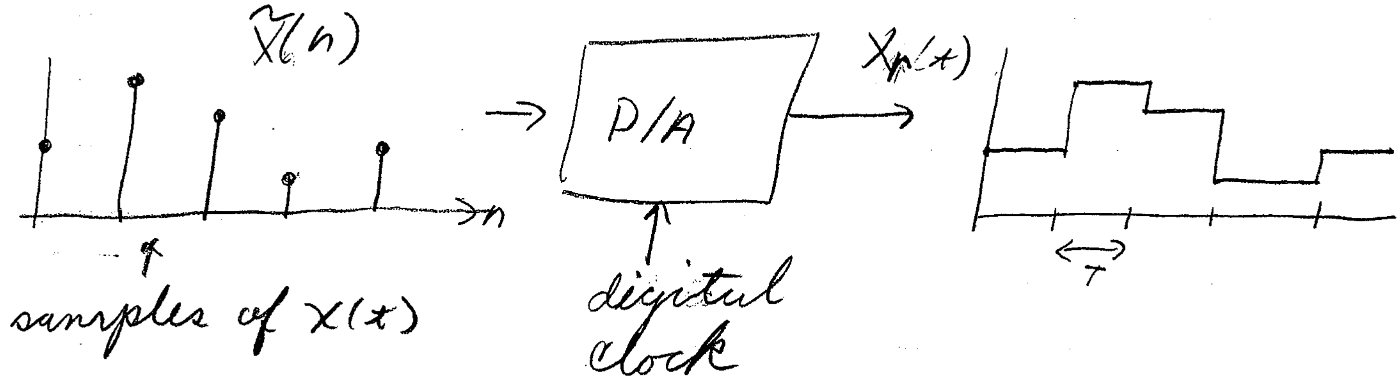
$$X_\delta(\omega) = \tilde{X}(\omega T) = \frac{1}{T}\sum_{K=-\infty}^{\infty} X\left(\omega + K\frac{2\pi}{T}\right)$$

$$\boxed{X_r(\omega) = \frac{P(\omega)}{T}\sum_{K=-\infty}^{\infty} X\left(\omega + K\frac{2\pi}{T}\right)}$$

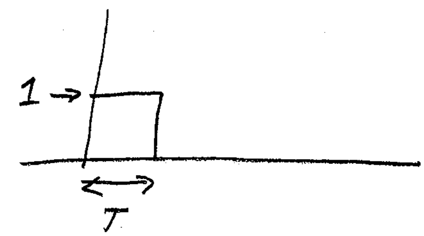
Real D/A converter

Digital to analog

<u>Example</u>



$$p(t) = u(t) - u(t-T)$$
$$= \mathrm{rect}((t - T/2)/T/2)$$

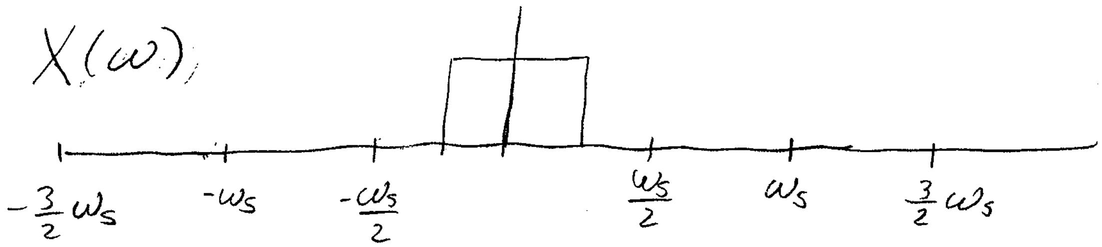


$$X_r(\omega) = \frac{P(\omega)}{T} \sum_{K=-\infty}^{\infty} X\left(\omega + K\frac{2\pi}{T}\right)$$

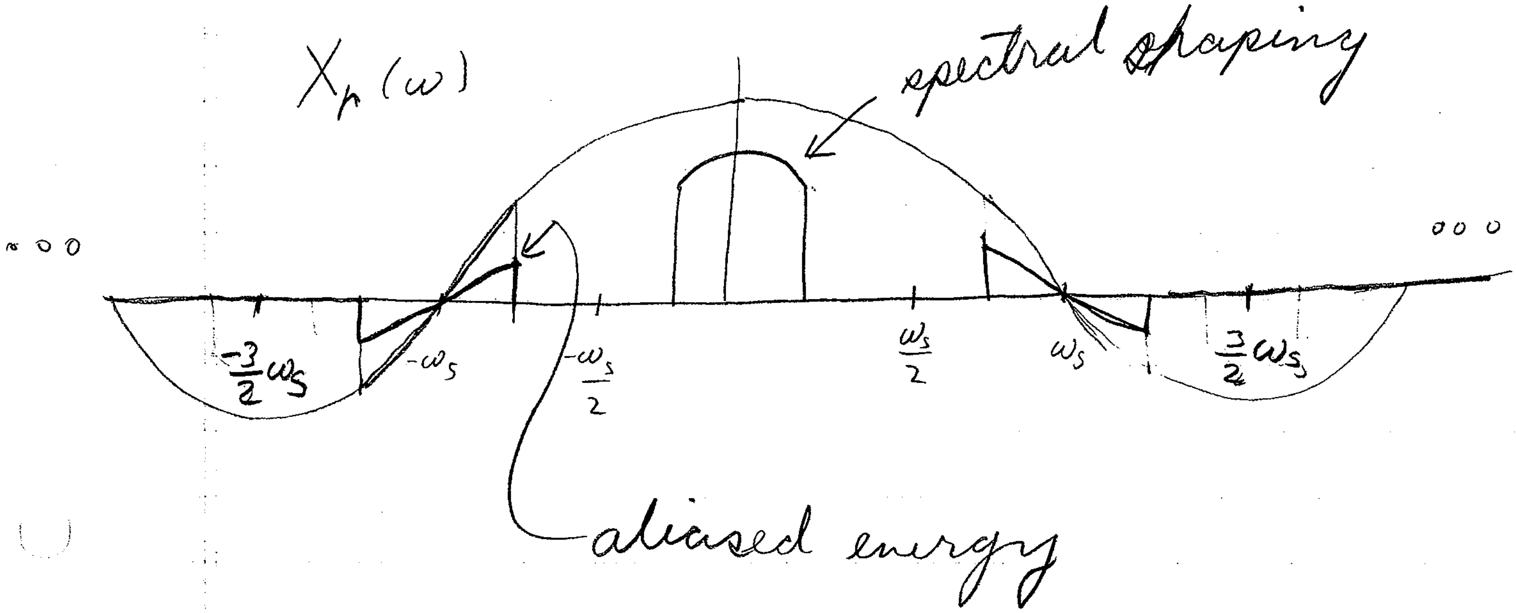
$$P(\omega) = \underbrace{e^{-j\omega T/2}}_{\text{pure delay of } T/2} T \,\mathrm{sinc}\left(\frac{\omega T}{2\pi}\right)$$

$$= e^{-j\omega T/2}\, T\,\mathrm{sinc}(\omega/\omega_s)$$

$$X_r(\omega) = e^{-j\omega T/2}\,\mathrm{sinc}(\omega/\omega_s) \sum_{K=-\infty}^{\infty} X\left(\omega + K\frac{2\pi}{T}\right)$$



X(ω)
$-\frac{3}{2}\omega_s$
$-\omega_s$
$-\frac{\omega_s}{2}$
$\frac{\omega_s}{2}$
ω_s
$\frac{3}{2}\omega_s$



$X_p(\omega)$
spectral shaping
$-\frac{3}{2}\omega_s$
$-\omega_s$
$-\frac{\omega_s}{2}$
$\frac{\omega_s}{2}$
ω_s
$\frac{3}{2}\omega_s$
aliased energy

Computing the approximate Fourier Transform of an signal using the DFT

$$\tilde{X}_K = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{x}(n) e^{-j\frac{2\pi}{N}kn}$$

$$\tilde{X}(\omega) = \sum_{n=-\infty}^{\infty} \tilde{x}(n)\, e^{-jn\omega}$$

If $\tilde{x}(n) = 0$ for $n < 0$ $n \geq N$

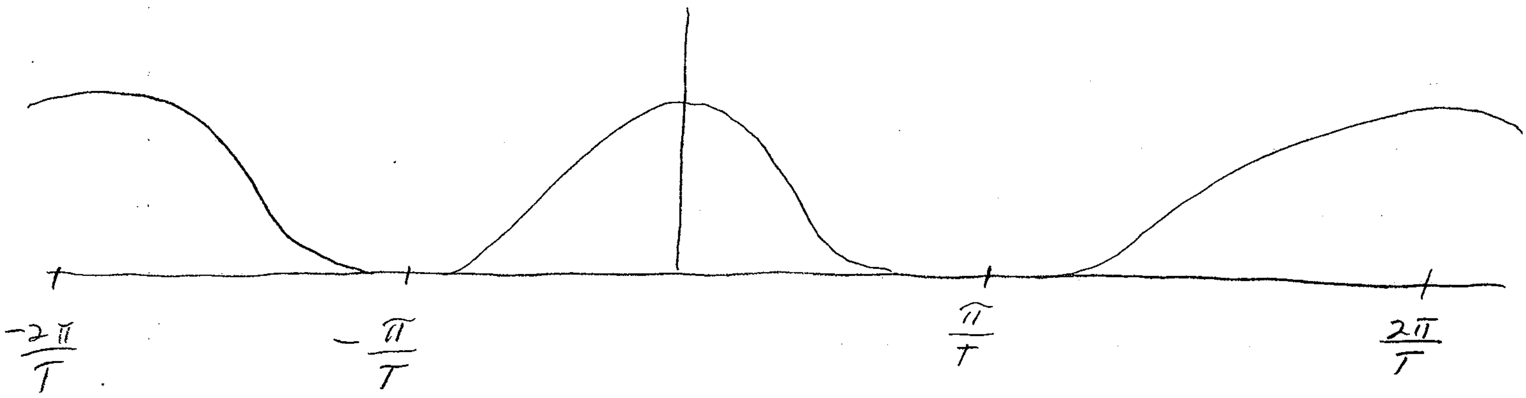
then $\tilde{X}_K = \frac{1}{N}\tilde{X}(\frac{2\pi}{N}k)$

But we want $X(\omega)$!

$$\tilde{X}(\Omega) = \frac{1}{T}\sum_{l=-\infty}^{\infty} X\left(\frac{\Omega + l2\pi}{T}\right)$$

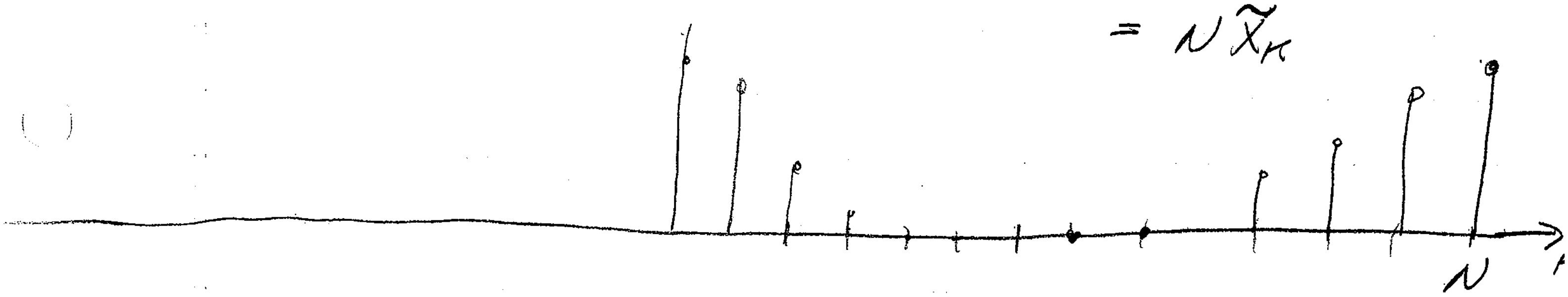
$$N\tilde{X}_K = \tilde{X}\left(\frac{2\pi K}{N}\right) = \frac{1}{T}\sum_{l=-\infty}^{\infty} X\left(k\frac{2\pi}{NT} + l\frac{2\pi}{T}\right)$$

$$\frac{1}{T}\sum_{l=-\infty}^{\infty} X\left(\omega + l\frac{2\pi}{T}\right)$$



If $\frac{\pi}{T}$ > maximum frequency

$$\frac{1}{T}\sum_{l=-\infty}^{\infty} X\left(k\frac{2\pi}{NT} + l\frac{2\pi}{T}\right) = N\tilde{X}_k$$



$$X\left(k\frac{2\pi}{NT}\right) \cong TN\tilde{X}_{k \bmod N} \quad \frac{-N}{2} \le k \; \frac{N}{2} - 1$$

Define $NT = D$ - duration of sampling

$$X\left(k\frac{2\pi}{D}\right) = TN\tilde{X}_{k \bmod N}$$

$\frac{2\pi}{D}$ is the "distance" between samples in ω

$\frac{2\pi}{T}$ is the separation between frequency "images" conclusion:

1) Increase D → fills in more points

2) Decrease T → Reduces alaising error.

Detail: what if $\tilde{x}(n) \neq 0 \quad n < 0$?

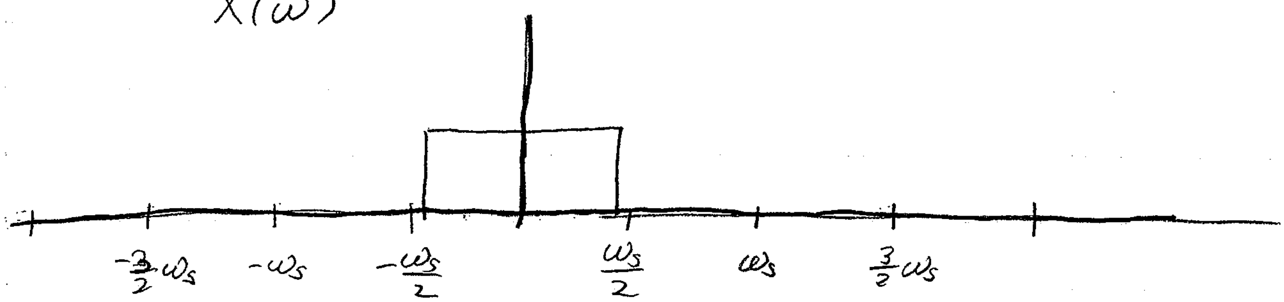
answer: extend $\tilde{x}(n)$ periodically

$$\tilde{X}_k = \sum_{n=0}^{N-1} \tilde{x}(n \bmod N)\, e^{-j\frac{2\pi}{N}kn}$$

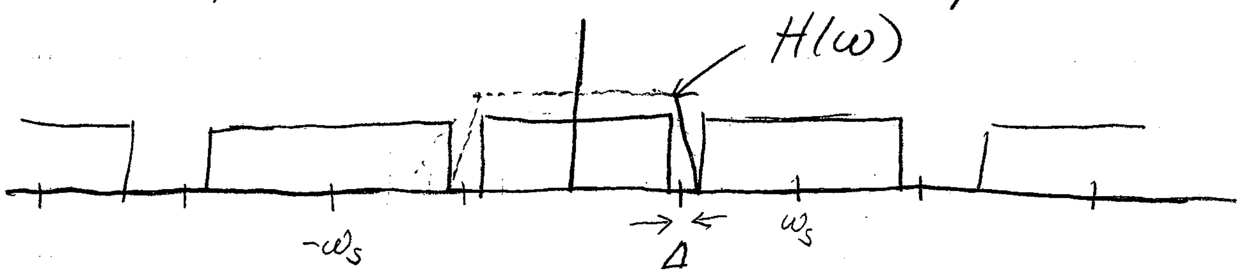
Question: What is 4× oversampling in a CD player?

$$X_r(\omega) = \sum_{K=-\infty}^{\infty} X\left(\omega + K\frac{2\pi}{T}\right)$$

$X(\omega)$



$X_p(\omega)$ – reconstruction with pulses



$\omega_s = 44.1$ KHz

$\frac{\omega_s}{2} = 22.05$

$\omega_M = 20$ KHz

$\Delta = 2(22.05 - 20) = 4.1$ KHz

$\frac{\Delta}{\omega_s/2} = .18594$

$\therefore H(\omega) < 2^{-16} \qquad \omega > \frac{\omega_s + \Delta}{2}$

$20 \log H(\omega) < -96.3 \cong -16(6) = -96$

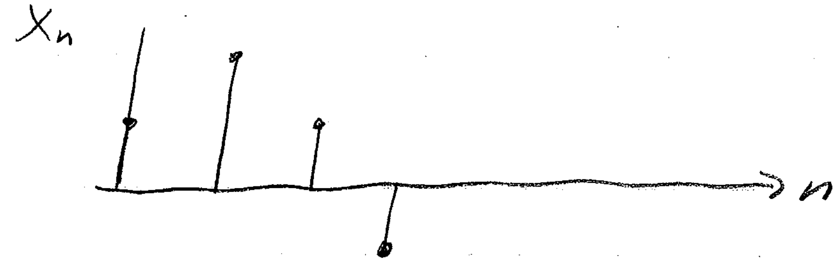
Can be done but expensive and difficult with analog components.

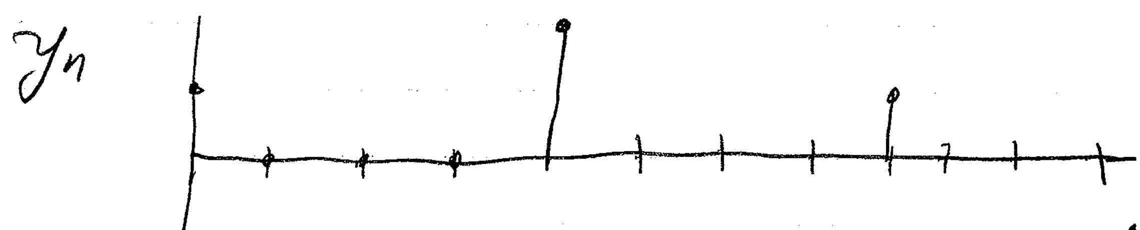
If ω_s was larger the job would be easyer.

Answer:

$$y_n = \begin{cases} x_{n/L} & \text{if } n/L \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}$$

$L = 4$ for $4\times$ oversampling



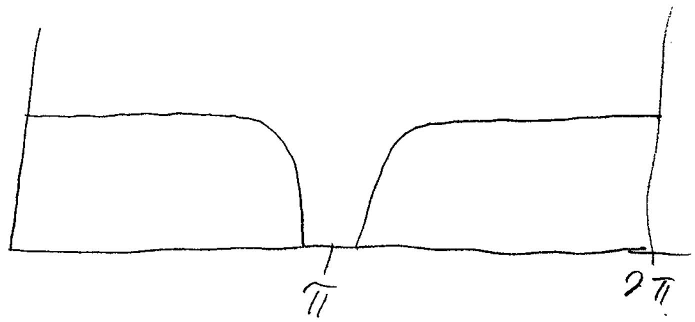


y_n is called and interpolated signal.

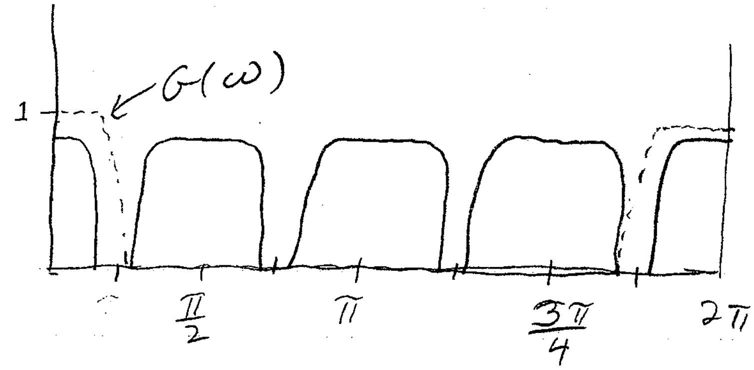
$$Y(\omega) = \sum_{n=-\infty}^{\infty} y_n e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x_n e^{j\omega L n} = X(L\omega)$$

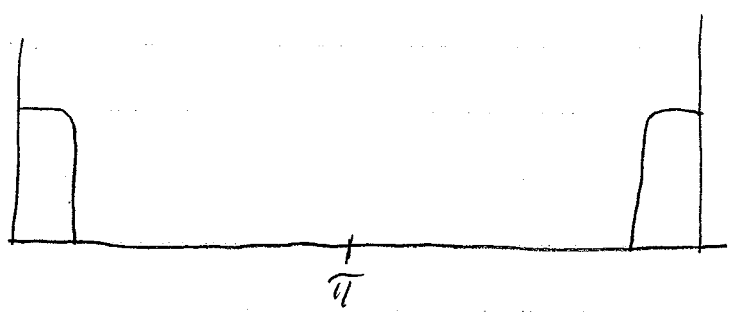
$X(\omega)$



$Y(\omega) = X(4\omega)$ $(L=4)$



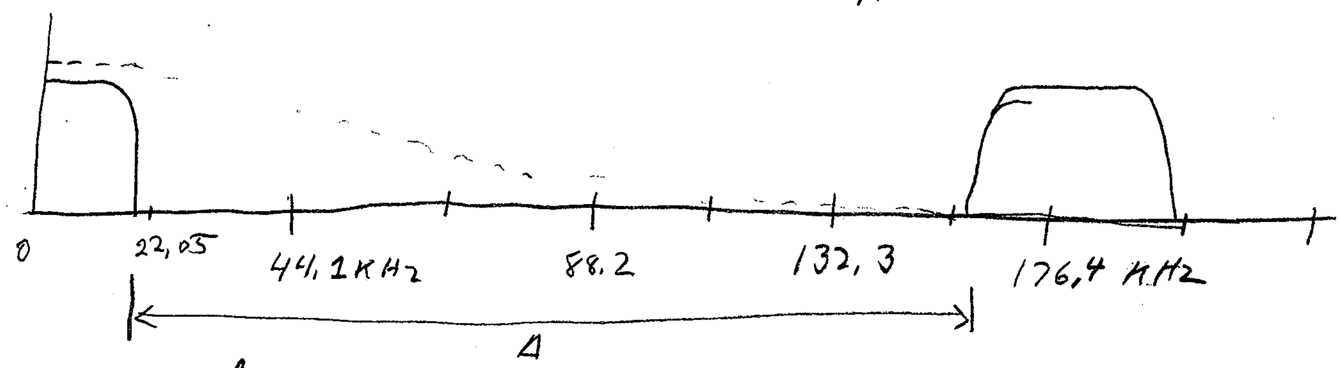
$G(\omega)\, Y(\omega)$



now use $\omega_s = 4(44.1\,KHz) = 176\,KHz$

$$X_p(\omega) = \sum_{K=-\infty}^{\infty} X(\omega + K\omega_s)$$

176 KHz



Now analog filtering is easy

$$\Delta = 3(22.05) + 2(22.05 - 20)$$
$$= 70.25 \text{ KHz}$$