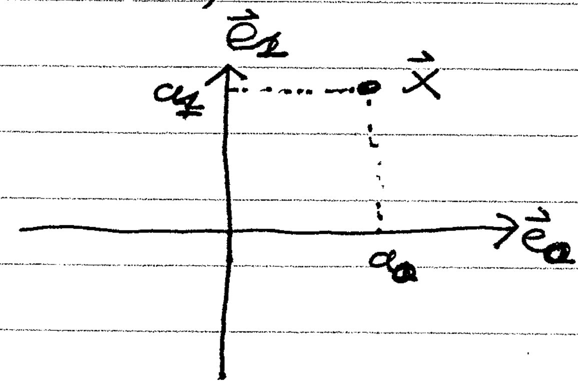
Orthonormal Transforms

Let X be a vector in a space with basies given by $\vec{e}_0, \vec{e}_2, \ldots, \vec{e}_{N-1}$

In 2-dimensions,



$$\vec{X} = a_1 \vec{e}_1 + a_2 \vec{e}_2$$

- $\vec{e}_0, \ldots, \vec{e}_{N-1}$ are known as orthonormal basis vectors if

1) For all K, $\|\vec{e}_K\|^2 = \vec{e}_K \cdot \vec{e}_K = 1$ (normal)

2) For all $K \neq l$ (orthogonal)
$$\vec{e}_K \cdot \vec{e}_l = 0$$

- Question! How do we compute the scalar values $a_0, \ldots, a_{N-1}$?

$$\vec{x} = \sum_{k=1}^{N} a_k \vec{e}_k$$

$$\vec{x} \cdot \vec{e}_\ell = \left(\sum_{k=1}^{N} a_k \vec{e}_k \right) \cdot \vec{e}_\ell$$

$$= \sum_{k=1}^{N} a_k \left(\vec{e}_k \cdot \vec{e}_\ell \right)$$

Notice that

$$\vec{e}_k \cdot \vec{e}_\ell = \begin{cases} 1 & \text{if } k=\ell \\ 0 & \text{if } k \neq \ell \end{cases}$$

$$= \delta_{k-\ell}$$

$$\vec{x} \cdot \vec{e}_\ell = \sum_{k=1}^{N} a_k \delta_{k-\ell}$$

$$= a_\ell$$

So $\quad a_\ell = \vec{x} \cdot \vec{e}_\ell$

Example:

$$\vec{x} = a_1 \vec{e}_0 + a_2 \vec{e}_1$$

$$a_0 = \vec{x} \cdot \vec{e}_0$$
$$a_1 = \vec{x} \circ \vec{e}_1$$

Exampl!

$$\vec{e}_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{e}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{e}_0 \cdot \vec{e}_0 = \frac{1}{2}(1+1) = 1$$

$$\vec{e}_1 \cdot \vec{e}_1 = \frac{1}{2}(1+1) = 1$$

$$\vec{e}_0 \cdot \vec{e}_1 = \frac{1}{2}(1-1) = 0$$

$$\vec{x} = a_0 \vec{e}_0 + a_2 \vec{e}_1$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$a_0 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \circ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(x_1 + x_2)$$

$$a_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \circ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(x_1 - x_2)$$

$$\vec{x} = \frac{1}{\sqrt{2}}(x_1 + x_2)\vec{e}_0 + \frac{1}{\sqrt{2}}(x_1 - x_2)\vec{e}_1$$

Functions as Vectors

- We can think of a complex ~~valued~~ valued function as a vector

So we have that the function

$\phi(x)$ is denoted as the vector ϕ. ~~We~~ I will write

$$\phi \Longleftrightarrow \phi(x)$$

Vector addition

$$f+g \Longleftrightarrow f(x)+g(x)$$

- Just add the functions

Vector inner product (dot product)

$$f \circ g = \langle f, g \rangle$$

$$\triangleq \int_a^b f(x)\, g^*(x)\, dx$$

complex conjugate

for some interval $[a, b]$