Properties of Fourier Series

P1 Linearity

Let $x(t) \longleftrightarrow a_k$

and

$y(t) \longleftrightarrow b_k$

then

$$\alpha x(t) + \beta y(t) \Longleftrightarrow \alpha a_k + \beta b_k$$

proof

Exercise

P2 Time Shift

Let $x(t) \leftrightarrow a_k$

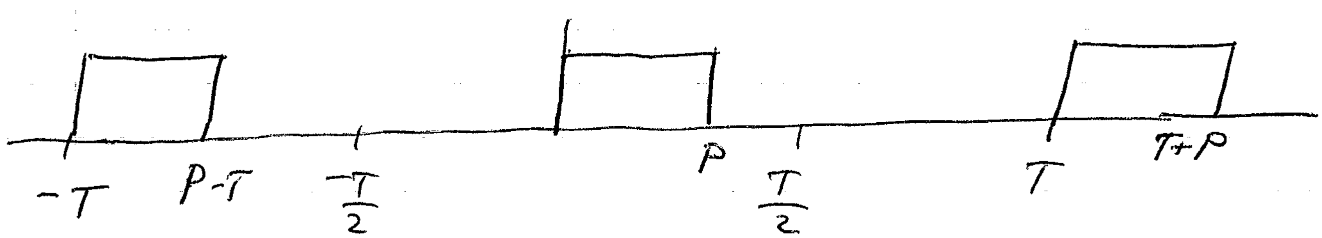
Then

$$x(t - t_0) \leftrightarrow a_k e^{-jk\omega t_0}$$

$$= a_k e^{-jk 2\pi t_0 / T}$$

proof Exercise

Example



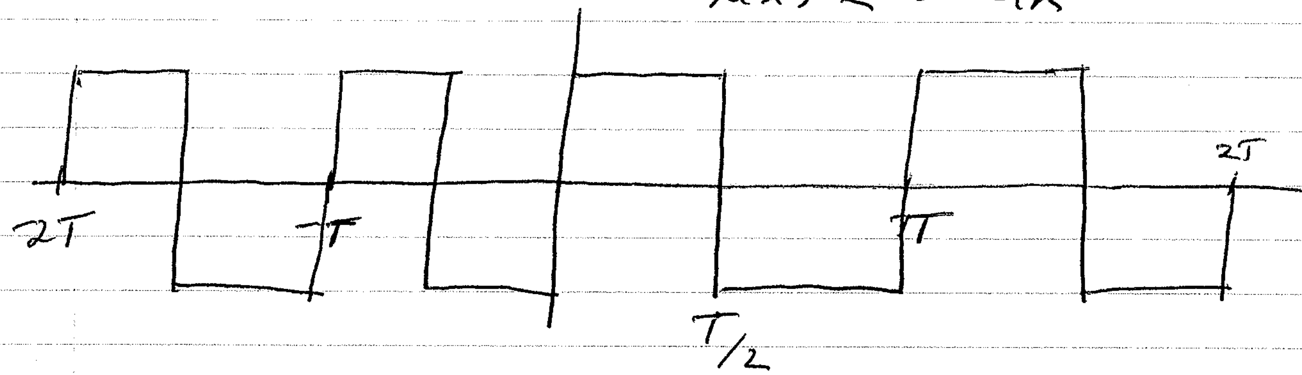
Same as before, but shifted by $P/2$

$$a_k = \frac{P}{T} \operatorname{sinc}(k P/T)\, e^{-j 2\pi (P/2) k / T}$$

$$= \frac{P}{T} \operatorname{sinc}(k P/T)\, e^{-j \pi k P / T}$$

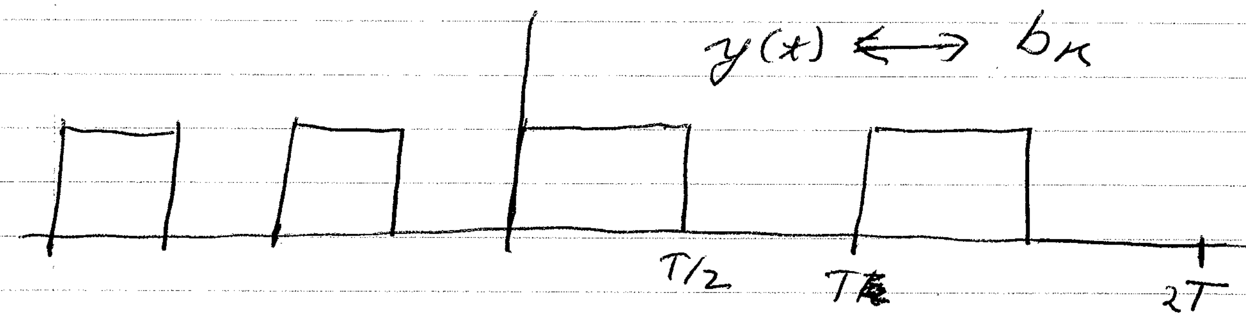
Example

$x(t) \Longleftrightarrow a_k$

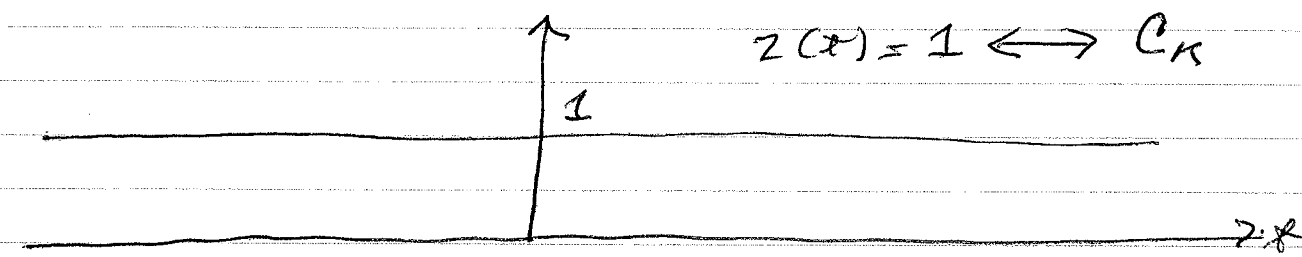


Use linearity

$y(t) \Longleftrightarrow b_k$



$z(t) = 1 \Longleftrightarrow c_k$



$$x(t) = 2y(t) - z(t)$$

We know that

$$y(t) \Longleftrightarrow b_k = \frac{p}{T} \operatorname{sinc}(k\, p/T)\, e^{-j\pi k p/T}$$

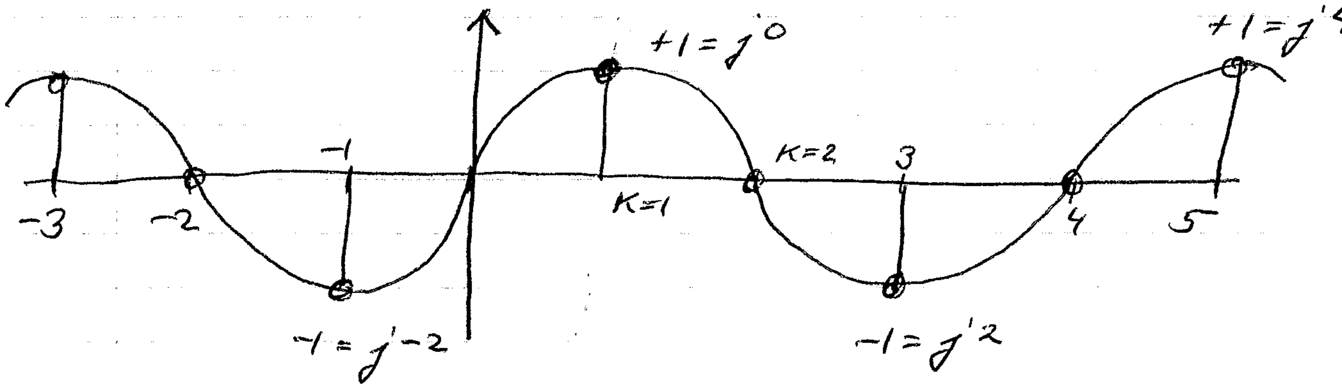
where $p = T/2$

$$b_k = \frac{1}{2} \operatorname{sinc}(k/2)\, e^{-j\pi k/2}$$

$$b_K = \frac{1}{2}\,\mathrm{sinc}(K/2)\,(-j)^K$$

Notice that

$$\mathrm{sinc}(K/2) = \frac{\sin(\pi K/2)}{\pi K/2}$$



So we have

$$\sin(\pi K/2) = \begin{cases} 0 & \text{for } K \text{ even} \\ (j)^{K-1} & \text{for } K \text{ odd} \end{cases}$$

So since $(-j)^K (j)^{K-1} = -j$

$$b_K = \begin{cases} \frac{1}{2} & \text{for } K=0 \\ 0 & \text{for } K \neq 0 \text{ and even} \\ \frac{-j}{\pi K} & \text{for } K \text{ odd} \end{cases}$$

Fourier Series for $z(t)$ is

$$z(t) \longleftrightarrow C_k = \delta[k]$$

Fourier series for $x(t)$ is

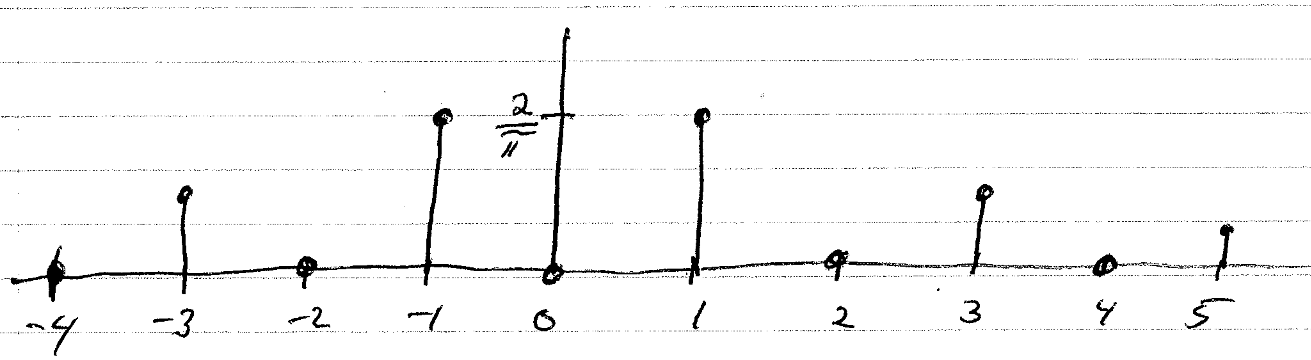
$$x(t) = 2y(t) - z(t)$$

$$a_k = 2b_k - c_k$$

$$a_k = 2b_k - \delta[k]$$

$$a_k = \begin{cases} 0 & \text{for } k \text{ even} \\ \dfrac{-2j}{\pi k} & \text{for } k \text{ odd} \end{cases}$$

$$|a_k| = \frac{2}{\pi k}$$



Comments

1) Only contains odd harmonics

2) $a_0 = 0$

$\Longleftrightarrow$ DC component $= 0$

$$\Longleftrightarrow \underbrace{\frac{1}{T}\int_{-T/2}^{T/2} x(t)\, 1\, dt}_{\text{average value}} = 0$$

3) Odd harmonics fall off as $\sim \frac{1}{k}$

Properties of Fourier Series

P3 Conjugate Symmetry

Assume $x(t)$ is real.

Then $a_K = a^*_{-K}$

proof:

$$a_K = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j\omega K t}\, dt$$

$$= \left(\frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{j K\omega t}\, dt\right)^*$$

$$= \left(\frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j(-K)\omega t}\, dt\right)^*$$

$$= \left(a_{-K}\right)^* = a^*_{-K}$$

P-4 Differentiation

Let $x(t) \overset{CTFS}{\Longleftrightarrow} a_k$

$$\frac{dx(t)}{dt} \overset{CTFS}{\Longleftrightarrow} jk\frac{2\pi}{T} a_k$$

proof: Homework

P-5 Time domain periodic convolution

$$z(t) = \int_{-T/2}^{T/2} x(\tau)y(t-\tau)d\tau \overset{CTFS}{\Longleftrightarrow} T a_k b_k$$

P-6 Time domain multiplication

$$x(t)\,y(t) \overset{CTFS}{\Longleftrightarrow} a_k * b_k$$

$$= \sum_{\ell=-\infty}^{\infty} a_\ell \, b_{k-\ell}$$

Parseval's Relation and Energy

P7 Parsevals Relation

Let $x(t) \Leftrightarrow a_k$

then

$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$\begin{pmatrix}\text{Time domain} \\ \text{Power/Energy}\end{pmatrix} = \begin{pmatrix}\text{Frequency} \\ \text{domain energy}\end{pmatrix}$$

proof

Consider Any orthonormal transform

$$x = \sum_{k=-\infty}^{\infty} a_k \phi_k$$

then

$$\langle x, x\rangle = \left\langle \sum_{k=-\infty}^{\infty} a_k \phi_k , \sum_{\ell=-\infty}^{\infty} a_\ell \phi_\ell \right\rangle$$

$$= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_k a_\ell^* \langle \phi_k, \phi_\ell \rangle$$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_k a_l^* T \delta[k-l]$$

$$= T \sum_{k=-\infty}^{\infty} |a_k|^2$$

<u>Notice</u>

$$\langle x, x \rangle = \int_{-T/2}^{T/2} |x(t)|^2 dt$$

<u>Comment</u>:

Factor of $1/T$ is due to the fact that Fourier series bases $e^{jk\omega t}$ are NOT normal.

Periodic Inputs to LTI Systems

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

where $x(t)$ is periodic with period $= T$

Then

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}$$

where $a_{-k} = a_k^*$ when $x(t)$ is real.

What is $y(t)$?

$$y(t) = S\left[x(t)\right]$$

$$= S\left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}\right]$$

$$= \sum_{k=-\infty}^{\infty} a_k S\left[e^{jk\omega t}\right]$$

Remember

$$e^{j(k\omega)t} \longrightarrow \boxed{h(t)} \longrightarrow C_k e^{j(k\omega)t}$$

where

$$C_k = \int_{-\infty}^{\infty} h(t) e^{-jk\omega t}\, dt$$

So we have that

$$y(t) = \sum_{k=-\infty}^{\infty} \underbrace{a_k c_k}_{b_k} e^{jk\omega t}$$

$b_k \triangleq a_k c_k \Leftarrow$ These are the Fourier series coefficients for $y(t)$!

<u>Time domain</u>

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = x(t) * h(t)$$

<u>Frequency domain</u>

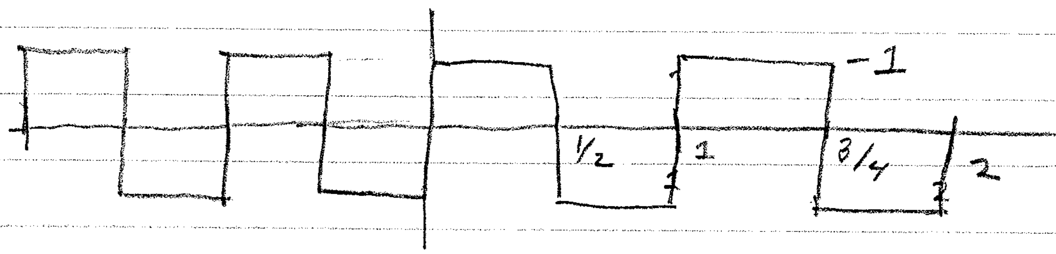
$$c_k \triangleq \int_{-\infty}^{\infty} h(t) e^{-j(k\omega t)} dt$$

$$a_k \rightarrow \boxed{c_k} \rightarrow b_k = a_k c_k$$

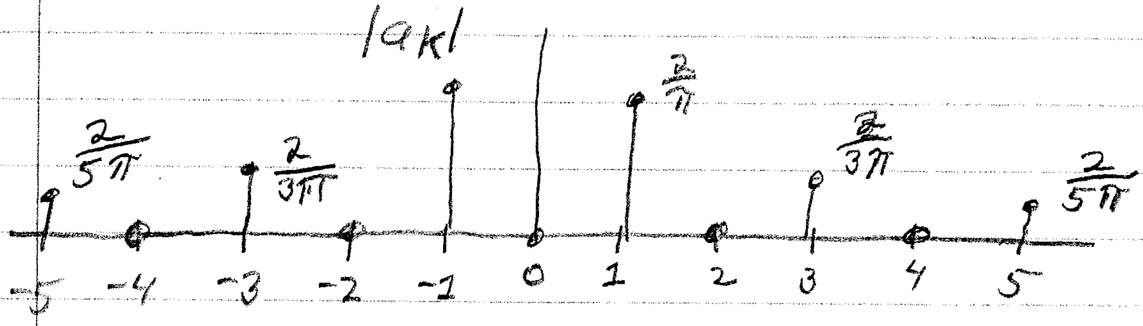
Conclusion: Frequency transform turns convolution into multiplication.

Example

$x(t) =$ 50% duty cycle square wave with period = 1



$$a_k = \begin{cases} 0 & \text{for } k \text{ even} \\ \dfrac{-2j}{\pi k} & \text{for } k \text{ odd} \end{cases}$$



$h(t)$ is Low Pass Filter (LPF)

$$C_k = \int_{-\infty}^{\infty} h(t)\, e^{-jk\omega t}\, dt$$

$$C_k = \begin{cases} 1 & \text{for } |k| \leq 2 \\ 0 & \text{for } |k| > 2 \end{cases}$$

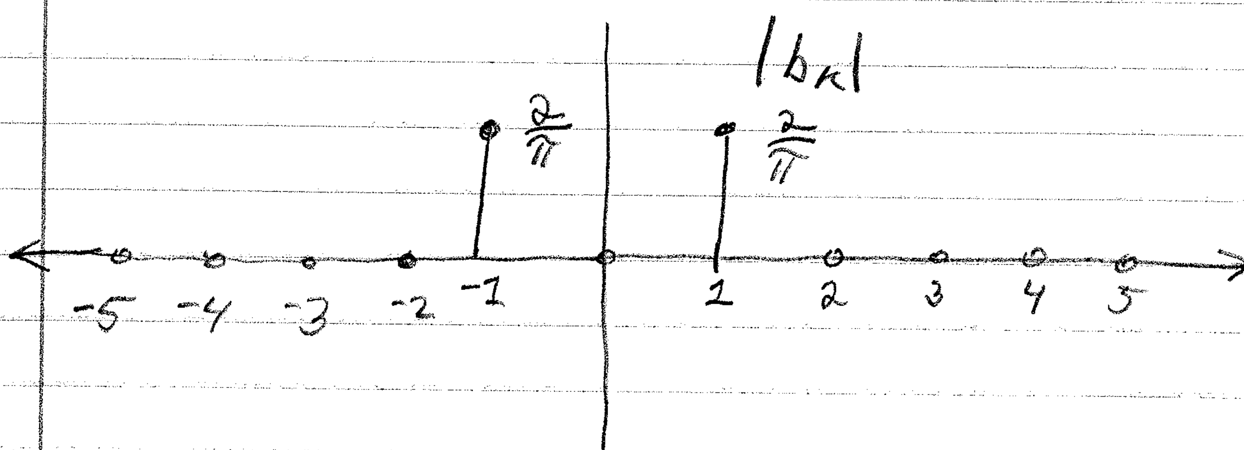
What is b_k?

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow \boxed{\begin{matrix} h(t) \\ C_k \end{matrix}} \longrightarrow y(t)=\sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

$$b_k = a_k C_k$$

k	0	1	2	3	4	5
a_k	0	$\frac{-2j}{\pi}$	0	$\frac{-2j}{3\pi}$	0	$\frac{-2j}{5\pi}$
C_k	1	1	1	0	0	0
b_k	0	$\frac{-2j}{\pi}$	0	0	0	0

$$b_k = \begin{cases} \frac{-2j}{\pi} & k=1 \\ \frac{2j}{\pi} & k=-1 \\ 0 & \text{otherwise} \end{cases}$$



What is $y(t) = ?$

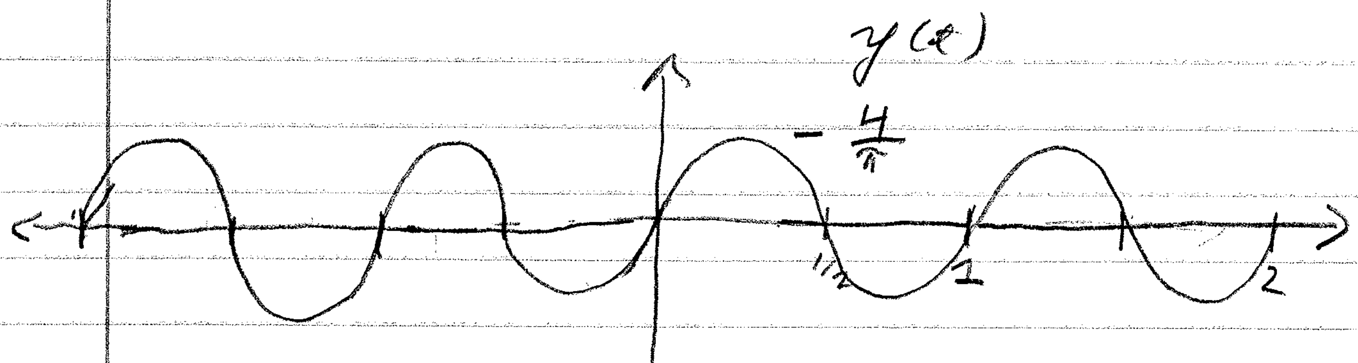
$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega t}$$

$$= -\frac{2j}{\pi} e^{j\omega t} + \frac{2j}{\pi} e^{-j\omega t}$$

$$= \frac{4}{\pi}\left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)$$

$$= \frac{4}{\pi} \sin(\omega t)$$

$$= \frac{4}{\pi} \sin(2\pi t)$$



What is the power of $y(t)$?

$$P_y = \frac{1}{T}\int_{-T/2}^{T/2} |y(t)|^2 \, dt$$

by Parsevals Relation we know that

$$P_y = \sum_{k=-\infty}^{\infty} |b_k|^2 = \sum_{k=-\infty}^{\infty} |a_k c_k|^2$$

$$= \sum_{k=-\infty}^{\infty} |a_k|^2 |c_k|^2$$

$$= \sum_{k=-K_c+1}^{K_c-1} |a_k|^2$$

$$= \sum_{k=-K_c+1}^{K_c-1} \left|\tfrac{1}{2}\operatorname{sinc}(k/2)\right|^2$$

$$= \underbrace{\left|\tfrac{1}{2}\right|^2}_{k=0} + \sum_{k=1}^{K_c-1} 2\left|\tfrac{1}{2}\operatorname{sinc}(k/2)\right|^2$$

$$= \frac{1}{4} + \sum_{\ell=0}^{\left\lfloor \frac{K_c-2}{2} \right\rfloor} \tfrac{1}{2}\left|\operatorname{sinc}((2\ell+1)/2)\right|^2$$

$$= \frac{1}{4} + \sum_{\ell=0}^{\lfloor \frac{K_c-2}{2} \rfloor} \frac{1}{2} \left| \frac{1}{\pi\left(\frac{2\ell+1}{2}\right)} \right|^2$$

$$= \frac{1}{4} + \frac{2}{\pi^2} \sum_{\ell=0}^{\lfloor \frac{K_c-2}{2} \rfloor} \frac{1}{(2\ell+1)^2}$$

What is the power of $x(t)$?

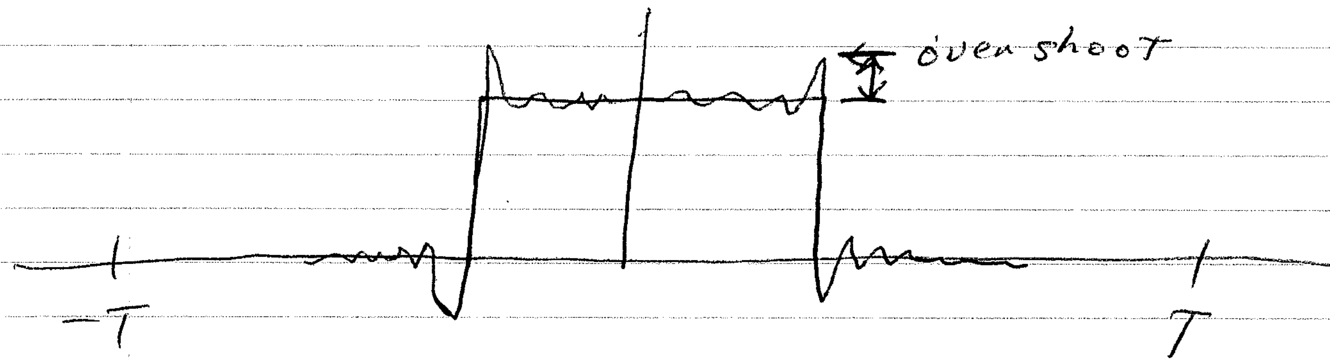
$$P_x = \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

So notice that $P_y < P_x$

$$P_y = \sum_{k=-K_c+1}^{K_c-1} |a_k|^2 < \sum_{k=-\infty}^{\infty} |a_k|^2 = P_x$$

$\Rightarrow$ Low pass filter reduces power.

What does $y(t)$ look like?



Gibbs Phenomenon:

1) Overshoot never goes to zero

2) Error energy goes to zero

Error Energy =

$$\frac{1}{T}\int_{-\frac{T}{2}}^{T/2} |x(t)-y(t)|^2\,dt$$

$$= \sum_{k=-\infty}^{\infty} |a_k - b_k|^2$$

$$= \sum_{k=-\infty}^{-k_c} |a_k|^2 + \sum_{k=k_c}^{\infty} |a_k|^2$$

$$\lim_{K_c \to \infty} \text{Error Energy} =$$

$$= \lim_{K_c \to \infty} \left\{ \sum_{k=-\infty}^{-K_c} |a_k|^2 + \sum_{k=K_c}^{\infty} |a_k|^2 \right\}$$

$$= 0$$

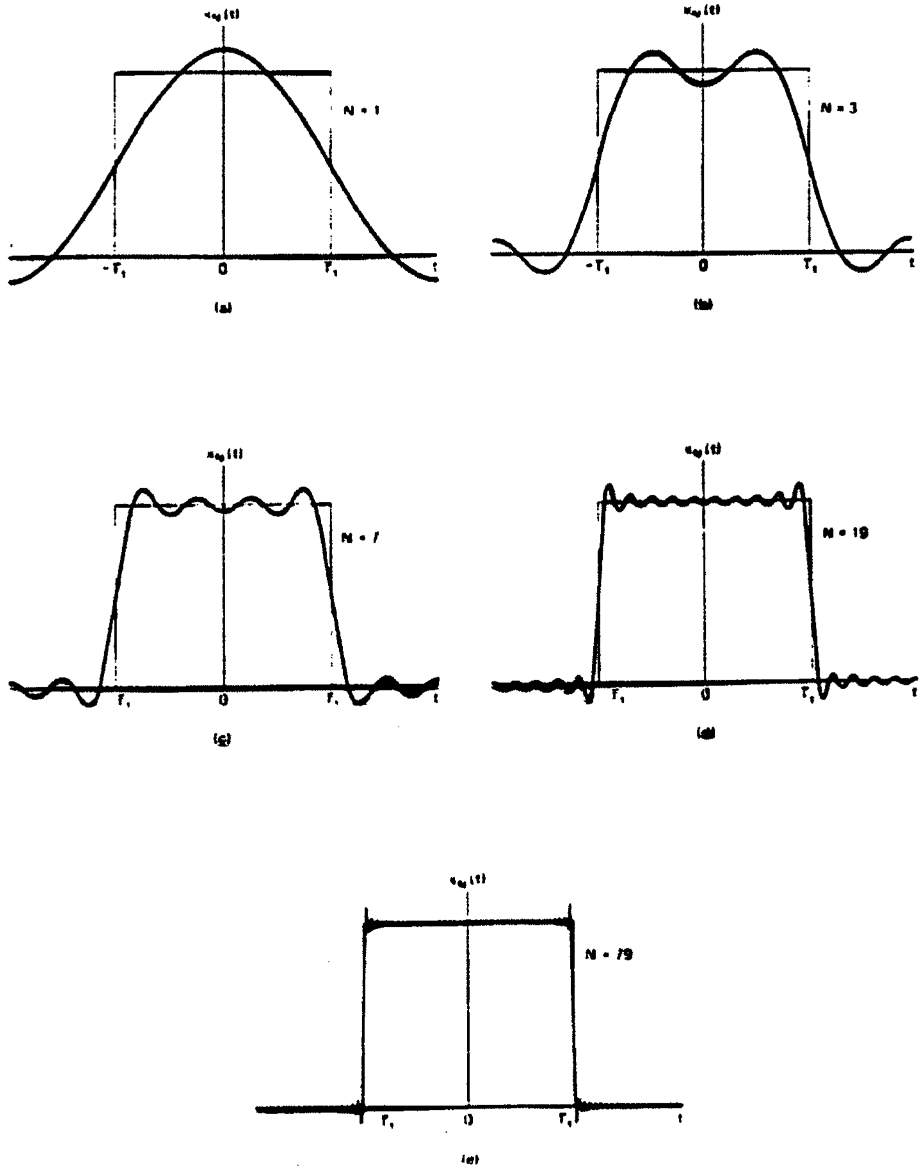


Figure 4.10 Convergence of the Fourier series representation of a square wave: an illustration of the Gibbs phenomenon. Here we have depicted the finite series approximation $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ for several values of N.