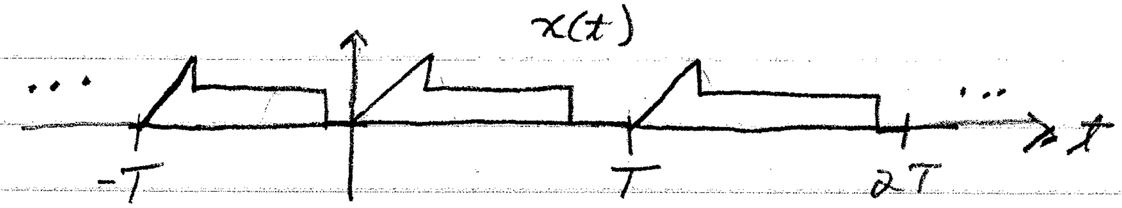
Fourier Series Concept

Let $x(t)$ be a periodic signal with period $T = \frac{2\pi}{\omega}$



We will represent $x(t)$ as sum of sin waves.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \underbrace{e^{jk\omega t}}_{\text{sin wave}}$$

Complex coefficient with amplitude and phase.

The sin wave basis functions are

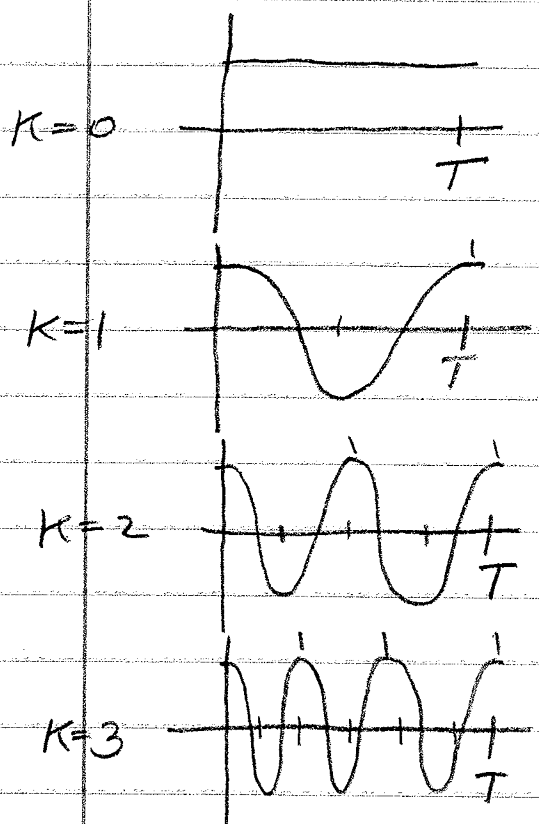
$$\phi_k(t) = e^{jk\omega t} \qquad \omega = \frac{2\pi}{T}$$

Notice that the ϕ_k are:

1) periodic with period T
2) periodic with period $\underbrace{kT}_{\text{fundamental period}}$

Intuitively

$$e^{jk\omega t} = \cos(k\omega t) + j\sin(k\omega t)$$



The ϕ_k's are basis vectors!

orthogonal:

$$\langle \phi_k, \phi_l \rangle = 0 \quad k \neq l$$

fixed length:

$$\|\phi_k\| = T$$

where

$$\langle x, y \rangle = \int_{-T/2}^{T/2} x(t)\, y^*(t)\, dt$$

Fourier Series Representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega t}$$

using vector notation

$$x = \sum_{k=-\infty}^{\infty} a_k \phi_k$$

basis vectors

what are the a_k's ?

$$\langle x, \phi_\ell \rangle = \langle \sum_{k=-\infty}^{\infty} a_k \phi_k , \phi_\ell \rangle$$

$$= \sum_{k=-\infty}^{\infty} a_k \langle \phi_k, \phi_\ell \rangle$$

$$= \sum_{k=\infty}^{\infty} a_k T \delta_{k-\ell}$$

$$= T a_\ell$$

$$a_\ell = \frac{1}{T} \langle x, \phi_\ell \rangle$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega t} dt$$

<u>proof</u> need to show that

$$\langle \phi_K, \phi_j \rangle \stackrel{?}{=} T\,\delta_{K-j}$$

$$\langle \phi_K, \phi_j \rangle = \int_{-T/2}^{T/2} e^{jK\omega t}\left(e^{j\ell\omega t}\right)^* dt$$

$$= \int_{-T/2}^{T/2} e^{j(K-\ell)\omega t}\, dt$$

$$= \frac{1}{j(K-\ell)\omega}\, e^{j(K-\ell)\omega t}\Big|_{-T/2}^{T/2}$$

$$= \frac{1}{j(K-\ell)\omega}\left(e^{j(K-\ell)\omega \frac{T}{2}} - e^{-j(K-\ell)\omega T/2}\right)$$

$$\omega = \frac{2\pi}{T} \Rightarrow \omega T = 2\pi$$

$$= \frac{T}{j(K-\ell)2\pi}\left(e^{j(K-\ell)\pi} - e^{-j(K-\ell)\pi}\right)$$

$$= \frac{T}{(K-\ell)\pi}\left(\frac{e^{j(K-\ell)\pi} - e^{-j(K-\ell)\pi}}{2j}\right)$$

$$= T\,\frac{\sin((K-\ell)\pi)}{(K-\ell)\pi}$$

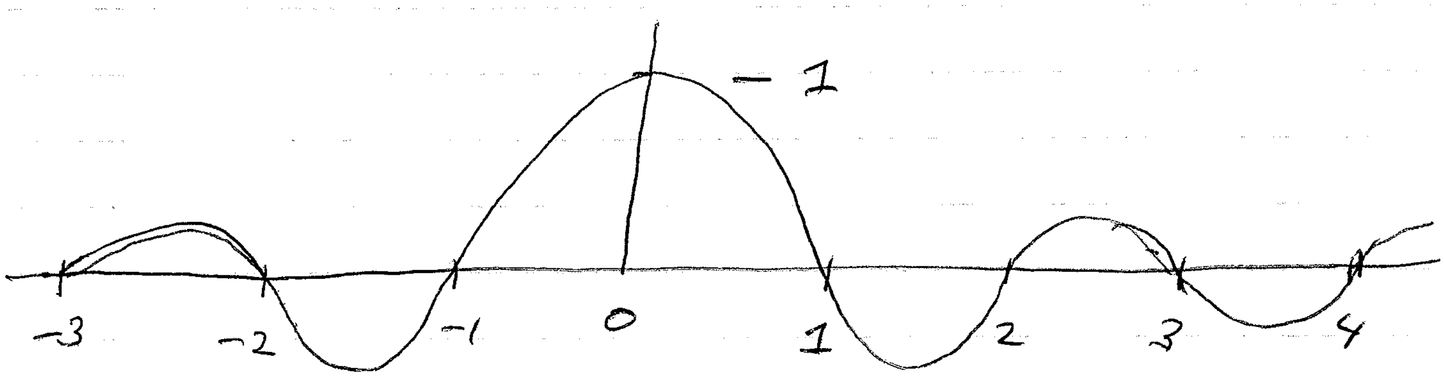
$$= \frac{T}{(k-l)\pi} \; \frac{e^{j(k-l)\pi} - e^{-j(k-l)\pi}}{2j}$$

$$= \frac{T}{(k-l)\pi} \sin((k-l)\pi)$$

$$= T\frac{\sin((k-l)\pi)}{(k-l)\pi} = T\,\mathrm{sinc}(k-l)$$

Definition:

$$\mathrm{sinc}(\theta) \triangleq \frac{\sin(\pi\theta)}{\pi\theta}$$



So we have that

$$\langle \phi_k, \phi_l \rangle = \begin{cases} T & \text{for } k=l \\ 0 & \text{for } k \neq l \end{cases}$$

Orthogonal!

Fourier Series Expansion

Transform

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, e^{-jk\omega t}\, dt$$

Inverse Transform

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{jk\omega t}$$

Comments

1) $x(t)$ is periodic with period $T = \frac{2\pi}{\omega}$

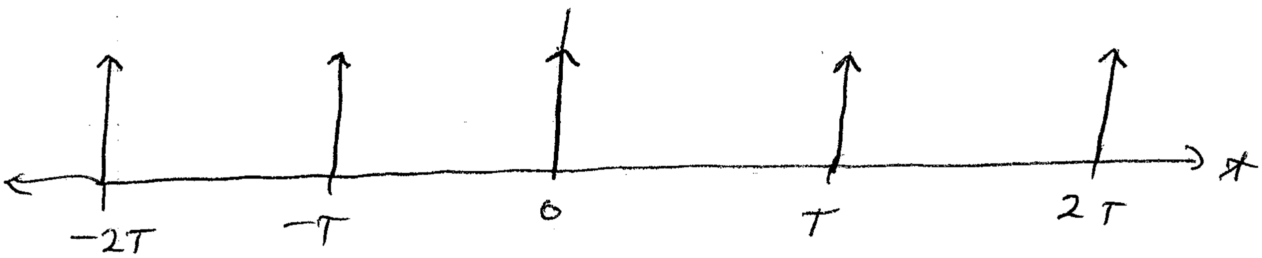
2) Integral may be computed over any single period

$$\int_{-\frac{T}{2}}^{T/2} x(t)\, e^{-jk\omega t}\, dt = \int_{t_0}^{t_0+T} x(t)\, e^{-jk\omega t}\, dt$$

for any value of t_0.

More Examples

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$



$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega t} \qquad \omega = \frac{2\pi}{T}$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jk\frac{2\pi}{T}t}\, dt$$

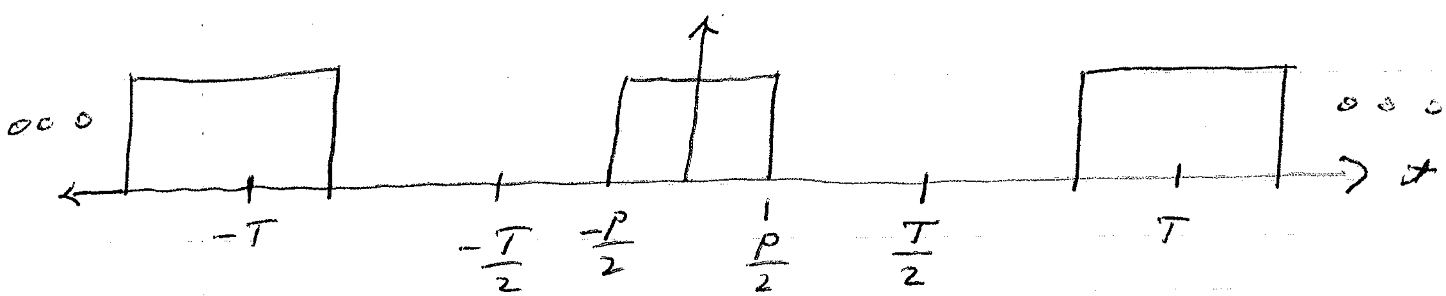
$$= \frac{1}{T} e^{-jk\frac{2\pi}{T}0} = 1/T$$

$$\boxed{a_k = 1/T}$$

Example

Let $x(t)$ be periodic with period T and

$$x(t) = \text{rect}(t/P) \quad \text{for } |t| < T/2$$



$$x(t) = \sum_{k=-\infty}^{\infty} \text{rect}\left(\frac{t-kT}{P}\right)$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\omega t}\, dt$$

where $\omega = \frac{2\pi}{T}$

$$= \frac{1}{T}\int_{-P/2}^{P/2} e^{-jk\omega t}\, dt$$

$$= \frac{1}{-jk\omega T} e^{-jk\omega t} \Big|_{-P/2}^{P/2}$$

Remember $\omega T = 2\pi$

$$= \frac{1}{-j2\pi k}\left(e^{-jk\omega P/2} - e^{+jk\omega P/2}\right)$$

$$= \frac{1}{-j2\pi k}\left(e^{-jk\pi \frac{P}{T}} - e^{+jk\pi \frac{P}{T}}\right)$$

$$= \frac{+1}{+2\pi k \cancel{j}}\, 2\cancel{j}\, \frac{\left(e^{+jk\pi \frac{P}{T}} - e^{-jk\pi \frac{P}{T}}\right)}{2j}$$

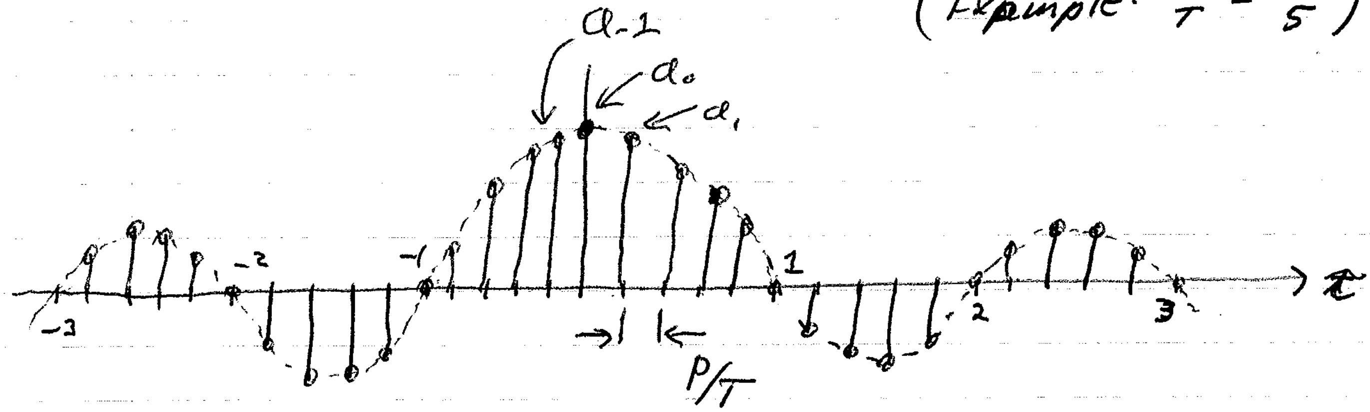
$$= \sin(\pi k P/T)\,\frac{1}{\pi k}$$

$$= \frac{P}{T}\,\frac{\sin(\pi k P/T)}{\pi k P/T}$$

$$= \frac{P}{T}\,\mathrm{sinc}(\underbrace{k P/T}_{\tau})$$

Notice that $P/T < 1$, so

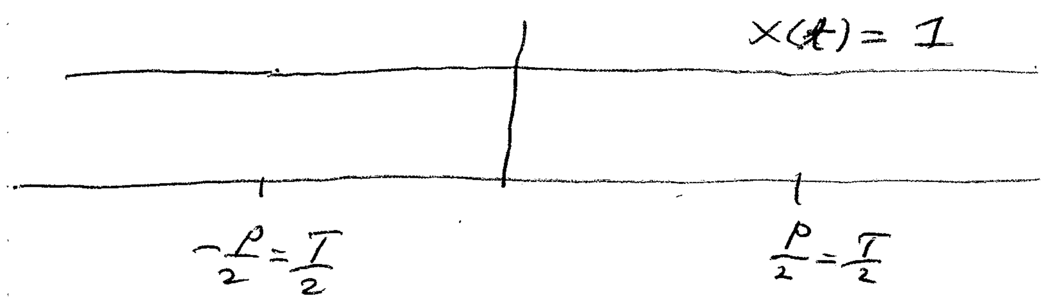
(Example: $\frac{P}{T} = \frac{1}{5}$)



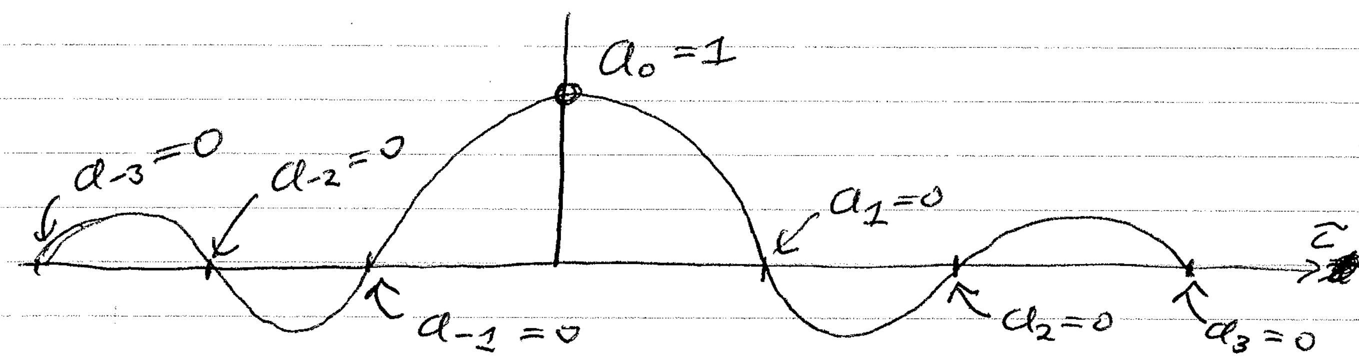
When P/T is small $\Rightarrow$
samples are close together

When P/T is large $\Rightarrow$
samples are far apart

<u>Special case $P = T$</u>



$a_0 = 1 \qquad a_k = 0$ for $k \neq 0$



$$a_k = \delta[k]$$

$$a_k = \frac{\sin(\pi k/2)}{k\pi}, \qquad k \neq 0 \tag{4.41}$$

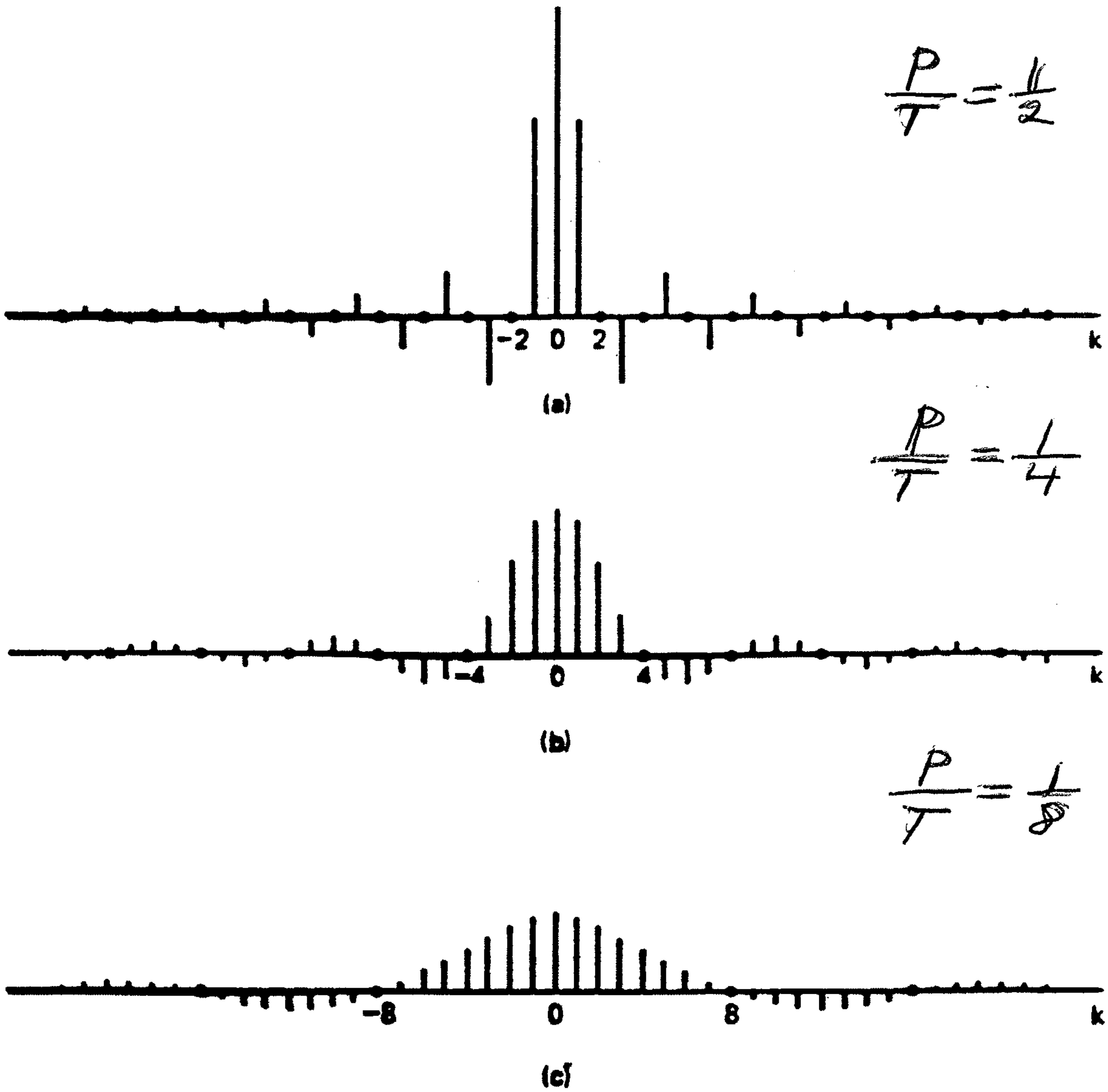


Figure 4.8 Fourier series coefficients for the periodic square wave: (a) $T_0 = 4T_1$; (b) $T_0 = 8T_1$; (c) $T_0 = 16T_1$.

Fourier Series as Sums of Sin waves

Let $x(t)$ be real valued.

Then it can be shown that

$$a_k = a_{-k}^* = c_k e^{j\theta_k}$$

So

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kt/T}$$

$$= a_0 + \sum_{k=1}^{\infty} \left(a_k e^{j2\pi kt/T} + a_k^* e^{-j2\pi kt/T} \right)$$

$$= a_0 + \sum_{k=1}^{\infty} c_k \left(e^{j(2\pi kt/T + \theta_k)} + e^{-j(2\pi kt/T + \theta_k)} \right)$$

$$= a_0 + \sum_{k=1}^{\infty} 2c_k \cos(2\pi kt/T + \theta_k)$$

Example

$$x(t) = \sum_{k=-\infty}^{\infty} \text{rect}\left(\frac{t-kT}{P}\right)$$

$$x(t) \overset{CTFS}{\Longleftrightarrow} a_k = \frac{P}{T}\,\text{sinc}(kP/T)$$

$$a_k = c_k e^{j\theta_k}$$

$$a_0 = \frac{P}{T}$$

$$c_k = \frac{P}{T}\,\text{sinc}(kP/T)$$

$$\theta_k = 0$$

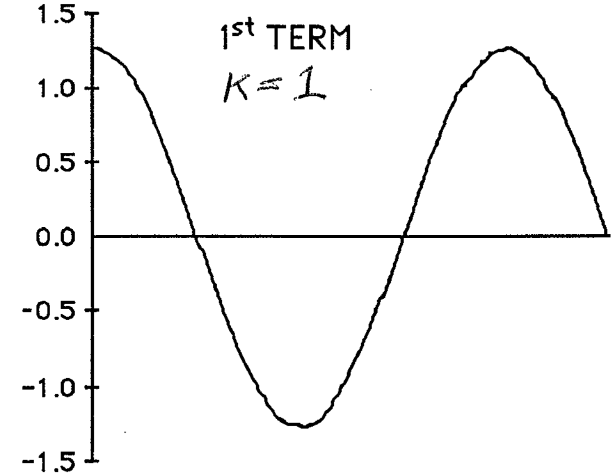
So

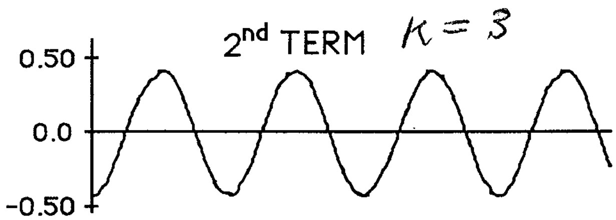
$$x(t) = \frac{P}{T} + \sum_{k=1}^{\infty} 2c_k \cos(2\pi k t/T)$$

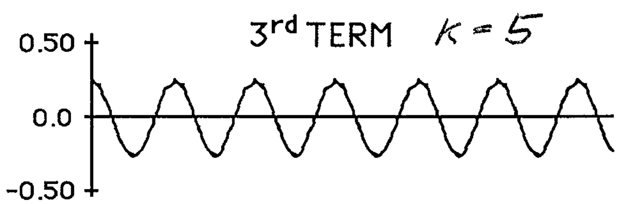
If $\frac{P}{T} = \frac{1}{2}$, then $c_k = 0$ for k even

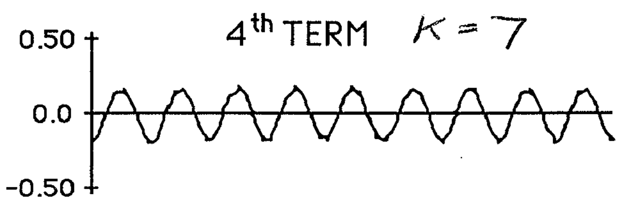
$\frac{P}{T} = \frac{1}{2}$

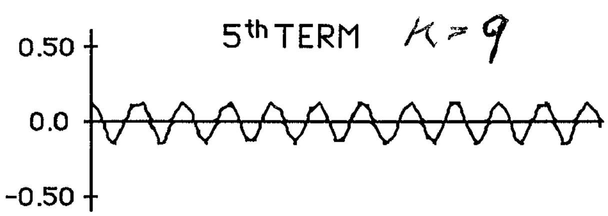
INDIVIDUAL TERMS | **PARTIAL SUMS**

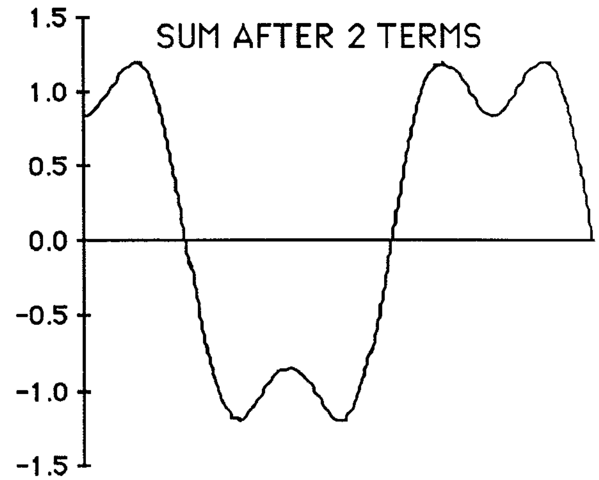


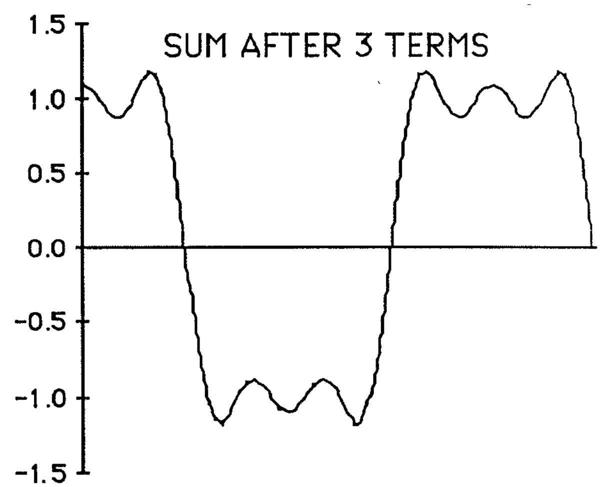


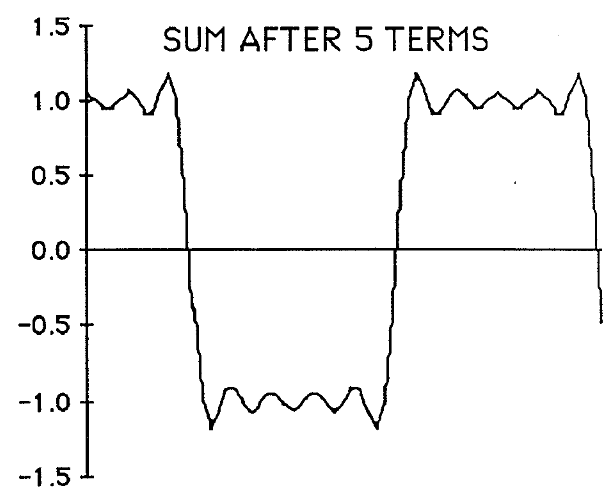












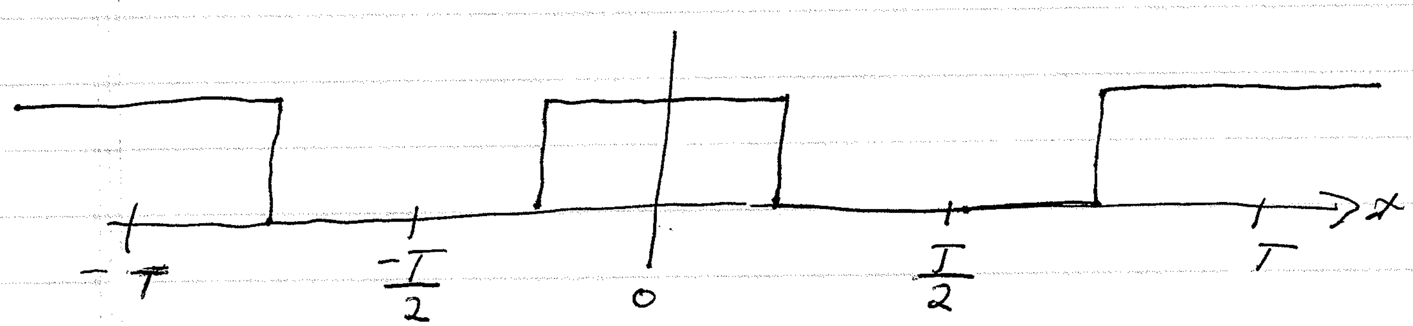
Example:

$x(t)$ periodic with period T

$$x(t) = \text{rect}\,(t/(T/2)) \quad |t| < T/2$$

$$x(t) = \text{rect}\,(2t/T) \quad |t| < T/2$$

$$x(t) = \sum_{k=-\infty}^{\infty} \text{rect}\left(\frac{2(t-kT)}{T}\right)$$



$$x(t) = \text{rect}\,(t/p) \quad |t| < T/2$$

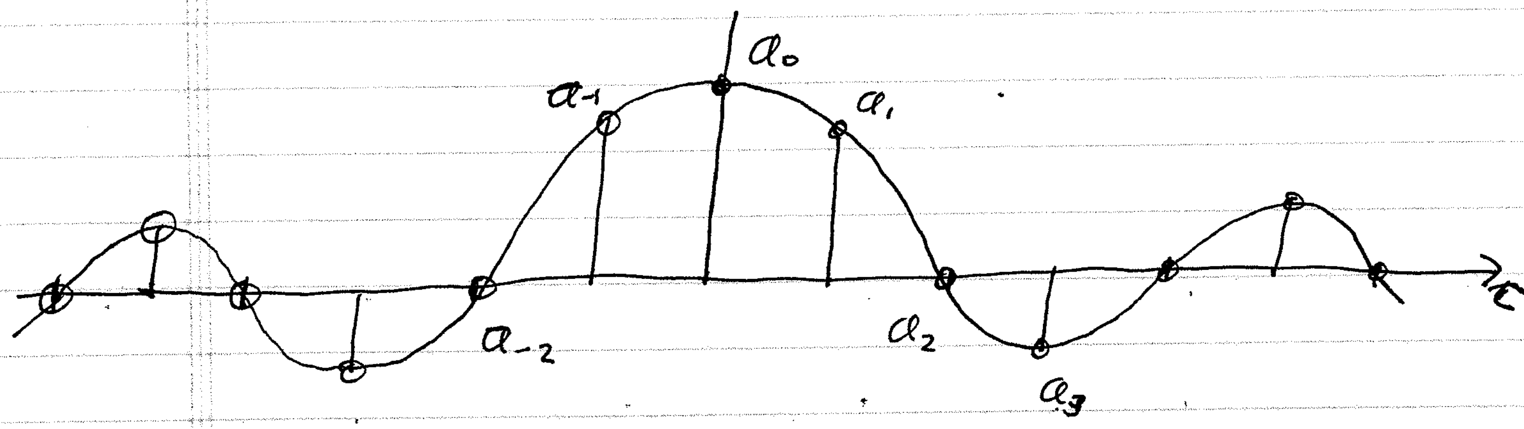
where $p = T/2$

So we know that

$$a_k = \frac{p}{T} \text{sinc}\,(kp/T)$$

$$a_k = \frac{1}{2} \text{sinc}\,(k/2)$$

$$a_K = \frac{1}{2} \operatorname{sinc}(\underbrace{K/2}_{t})$$



So we see that

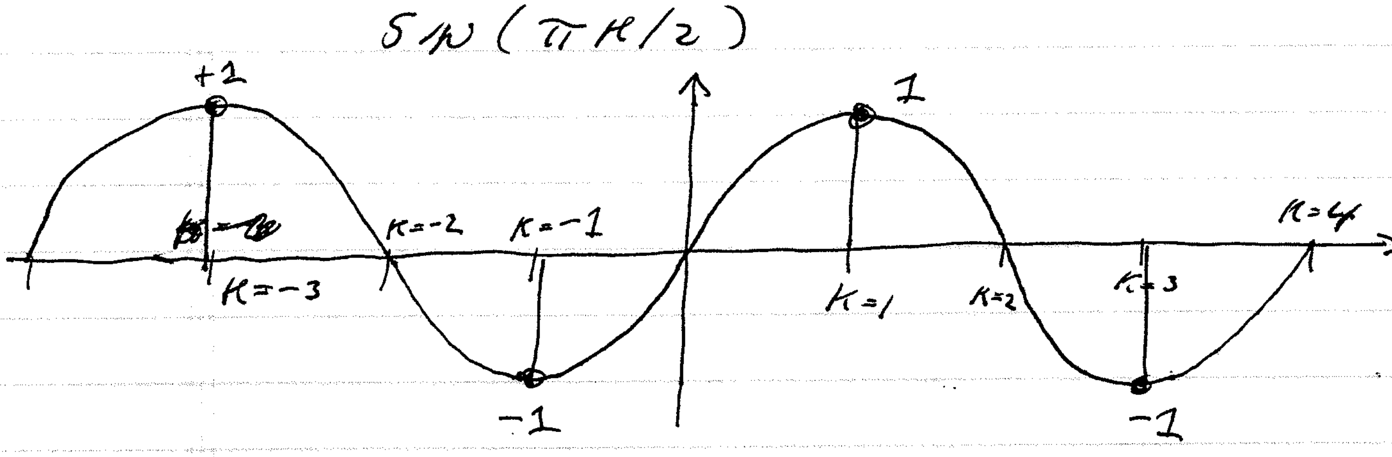
$a_0 = 1/2$

$a_K = 0$ for K even and $\neq 0$

$a_K = \frac{1}{2} \operatorname{sinc}(K/2)$ for K odd

$$\frac{1}{2} \operatorname{sinc}(K/2) = \frac{1}{2} \frac{\sin(\pi K/2)}{\pi K/2}$$

$$\frac{1}{2}\,\mathrm{sinc}(k/2) = \frac{1}{2}\,\frac{\sin(\pi k/2)}{\pi k/2}$$



So we have for <u>k odd</u>

$$\sin(\pi k/2) = (-1)^{(k-1)/2}$$

$$a_k = \frac{1}{2}\,\mathrm{sinc}(k/2) = \frac{(-1)^{(k-1)/2}}{\pi k}$$

So for all ~~k~~ we have

$$a_k = \begin{cases} 1/2 & \text{for } k=0 \\ 0 & \text{for } k \text{ even } (\neq 0) \\ \dfrac{(-1)^{(k-1)/2}}{\pi k} & \text{for } k \text{ odd} \end{cases}$$

Plot of $|a_k|$