The Discrete Time Fourier Transform (DTFT)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

Facts

1) $X(\omega)$ is periodic with period 2π.

2) So we also know that

$$x[n] = \frac{1}{2\pi} \int_{0}^{2\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

3) Discrete in time, but continuous in frequency.

Properties of DTFT

P1 Linearity

$$a x_1(n) + b x_2(n) \leftrightarrow a X_1(\omega) + b X_2(\omega)$$

P2. Time Reversal

$$x(-n) \leftrightarrow X(-\omega)$$

P3. Frequency Symmetry

If $x(n)$ is real:

$$X(\omega) = X^*(-\omega)$$

P4. Time shift

$$x(n - \underset{\substack{\uparrow \\ \text{integer}}}{T}) \leftrightarrow e^{-j\omega T} X(\omega)$$

P5. Frequency Shift

$$x(n) e^{j\omega_0 n} \leftrightarrow X(\omega - \omega_0)$$

P5 Convolution

$$x_n * h_n \overset{DTFT}{\Longleftrightarrow} X(\omega)\, H(\omega)$$

P7 Modulation

$$x_n\, h_n \overset{DTFT}{\Longleftrightarrow} \frac{1}{2\pi}\, X(\omega) \circledast H(\omega)$$

periodic convolution

$$X(\omega) \circledast H(\omega) = \int_0^{2\pi} X(\gamma)\, H(\omega-\gamma)\, d\gamma$$

P8 Parseval's Relation

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega$$

Example

$$x_n = a^n u(n) \qquad |a| < 1$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} a^n u(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} (a e^{-j\omega})^n u(n)$$

$$= \sum_{n=0}^{\infty} (a e^{-j\omega})^n = \frac{1}{1 - a e^{-j\omega}}$$

Example

$$x[n] = 1$$

$$X(\omega) = 2\pi\,\delta(\omega) \qquad \text{for } |\omega| < \pi$$

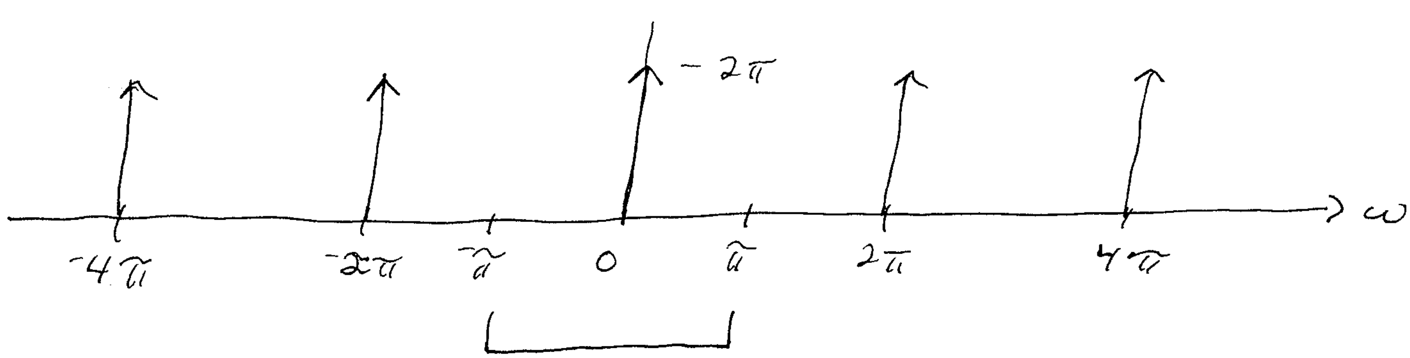
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{-j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\,\delta(\omega)\, e^{-j\omega n}\, d\omega$$

$$= 1$$

Remember, since DTFT is periodic with period 2π, we know

$$X(\omega) = \sum_{l=-\infty}^{\infty} 2\pi\,\delta(\omega - 2\pi l)$$



↑ frequency range of interest

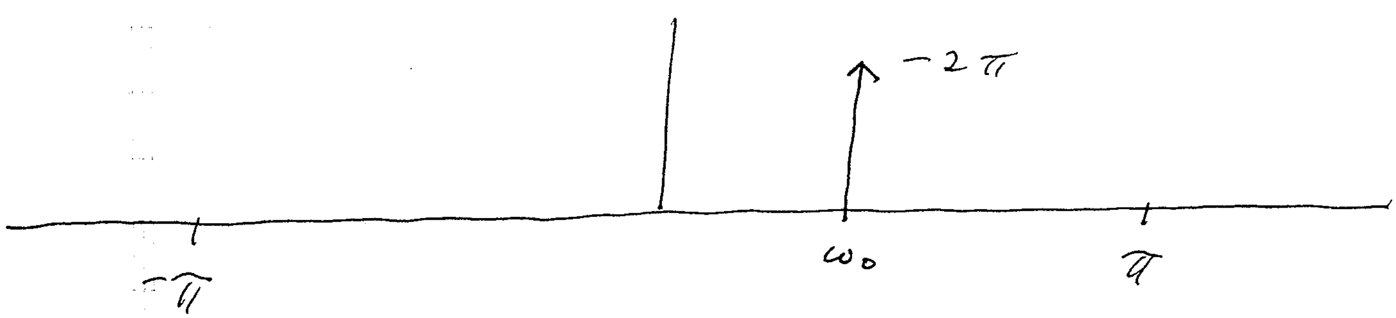
Example

$$x[n] = e^{j\omega_0 n}$$

By property (P5) we have

$$x[n] = 1 \cdot e^{j\omega_0 n}$$

$$x[n] \overset{DTFT}{\Longleftrightarrow} \sum_{\ell=-\infty}^{\infty} 2\pi\,\delta(\omega - \omega_0 - 2\pi\ell)$$

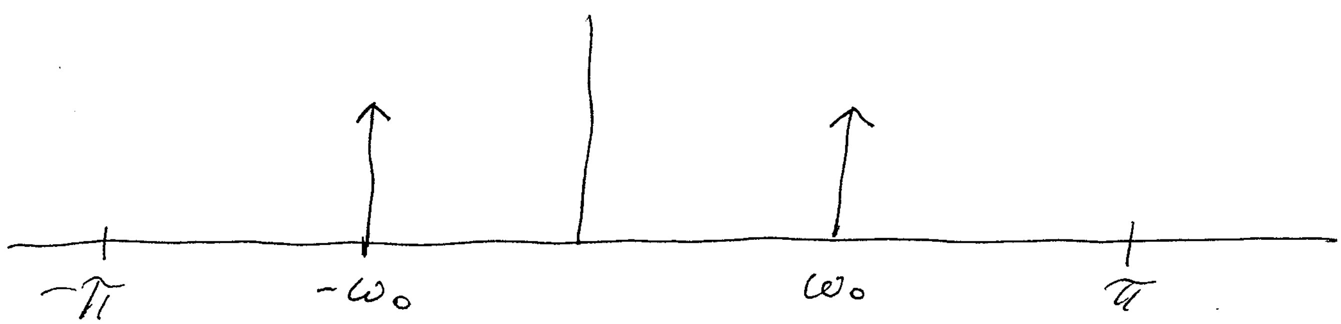


Example

$$x[n] = \cos(\omega_0 n)$$

$$= \frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}$$

$$x[n] \overset{DTFT}{\Longleftrightarrow} \frac{1}{2} \sum_{l=-\infty}^{\infty} 2\pi \left\{ \delta(\omega + \omega_0 + 2\pi l) - \delta(\omega - \omega_0 + 2\pi l) \right\}$$



Example

$$x[n] = u(n) - u(n-P)$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \sum_{n=0}^{P-1} e^{-j\omega n}$$

$$= \frac{1 - e^{-j\omega P}}{1 - e^{-j\omega}}$$

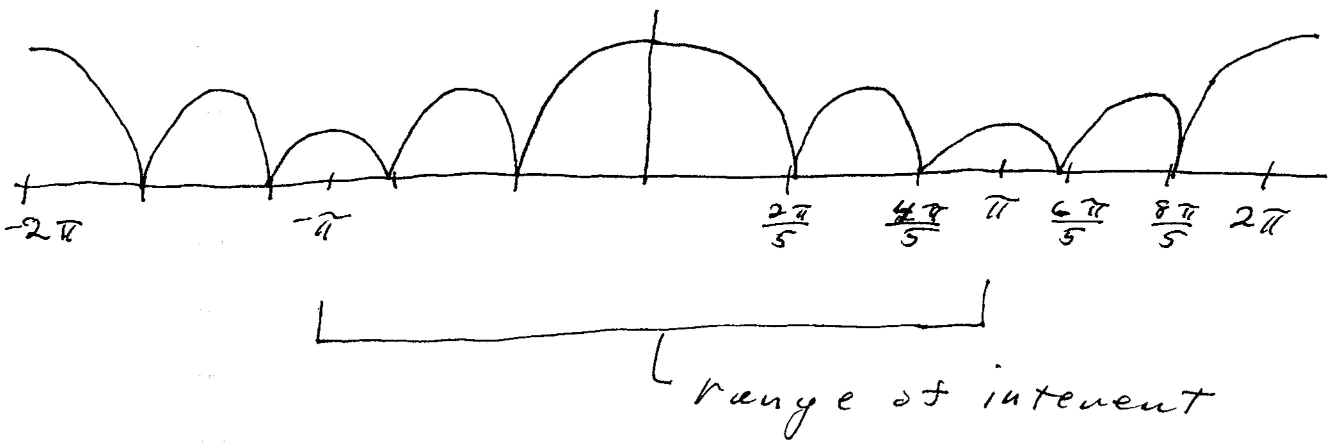
$$= \frac{e^{-j\omega P/2}}{e^{-j\omega/2}} \; \frac{e^{+j\omega P/2} - e^{-j\omega P/2}}{e^{+j\omega/2} - e^{-j\omega/2}}$$

$$= e^{-j\omega(P-1)/2} \; \frac{\sin(\omega P/2)}{\sin(\omega/2)}$$

$$= \underbrace{e^{-j\omega(P-1)/2}}_{\text{delay of } (P-1)/2} \; P\,\mathrm{sinc}_P(\omega/2\pi)$$

$$|X(\omega)| = |p\,\mathrm{sinc}_p(\omega/2\pi)|$$

$p = 5$



DTFT PAIRS

$$x[n] = 1 \overset{DTFT}{\Longleftrightarrow} X(\omega) = \sum_{\ell=-\infty}^{\infty} 2\pi\,\delta(\omega - 2\pi\ell)$$

$$x[n] = e^{j\omega_0 n} \overset{DTFT}{\Longleftrightarrow} X(\omega) = \sum_{\ell=-\infty}^{\infty} 2\pi\,\delta(\omega - \omega_0 - 2\pi\ell)$$

$$x[n] = \cos(\omega_0 n) \overset{DTFT}{\Longleftrightarrow}$$

$$X(\omega) = \sum_{\ell=-\infty}^{\infty} \pi \left\{ \delta(\omega + \omega_0 - 2\pi\ell) + \delta(\omega - \omega_0 - 2\pi\ell) \right\}$$

$$x[n] = a^n u[n]$$

$$\overset{DTFT}{\Longleftrightarrow} \quad \frac{1}{1 - a e^{-j\omega}}$$

$$x[n] = u(n) - u(n - p)$$

$$\overset{DTFT}{\Longleftrightarrow} \quad X(\omega) = e^{-j\omega\left(\frac{p-1}{2}\right)}\, p\,\mathrm{sinc}_p(\omega/2\pi)$$

DTFT's and LTI systems

i.e. DALS

$$x_n \longrightarrow \boxed{h_n} \longrightarrow y_n$$

$$y_n = \sum_{k=-\infty}^{\infty} x_k h_{n-k}$$

$$Y(\omega) = \sum_{n=-\infty}^{\infty} y_n e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_k h_{n-k} e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x_k \sum_{n=-\infty}^{\infty} h_{n-k} e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x_k e^{-j\omega k} \sum_{n=-\infty}^{\infty} h_{n-k} e^{-j\omega(n-k)}$$

$$= \sum_{k=-\infty}^{\infty} x_k e^{-j\omega k} \sum_{n=-\infty}^{\infty} h_n e^{-j\omega n}$$

$$= X(\omega)\, H(\omega)$$

Example: $x_n \rightarrow \boxed{h_n} \rightarrow y_n$

$$y_n = x_n * h_n$$

$$x_n = a^n u(n) \qquad |a| < 1 \qquad a \neq b$$
$$h_n = b^n u(n) \qquad |b| < 1$$

$$H(\omega) = \frac{1}{1 - b\, e^{-j\omega}}$$

$$X(\omega) = \frac{1}{1 - a\, e^{-j\omega}}$$

$$Y(\omega) = X(\omega) H(\omega) =$$

$$= \frac{1}{1 - a\, e^{-j\omega}} \frac{1}{1 - b\, e^{-j\omega}}$$

$$= \frac{A}{1 - a e^{-j\omega}} + \frac{B}{1 - b\, e^{-j\omega}}$$

$$A = \frac{a}{a-b} \qquad B = -\frac{b}{a-b}$$

$$= \left(\frac{a}{a-b}\right) \frac{1}{1 - a\, e^{-j\omega}} - \frac{b}{a-b} \frac{1}{1 - b e^{-j\omega}}$$

$$y_n = \frac{a}{a-b} a^n u(n) - \frac{b}{a-b} b^n u(n)$$

Discrete Time Systems modeled by difference Equations

$$(a_0 = 1) \quad \sum_{K=0}^{N} a_K\, y_{n-K} = \sum_{K=0}^{M} b_K\, x_{n-K}$$

(DTFT)

$$\sum_{K=0}^{N} a_K\, e^{-j\omega K}\, Y(\omega) = \sum_{K=0}^{M} b_K\, e^{-j\omega K}\, X(\omega)$$

$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = \frac{\sum_{K=0}^{N} b_K e^{-j\omega K}}{\sum_{K=0}^{N} a_K e^{-j\omega K}}$$

$$h(t) = DFT^{-1}\left\{ \frac{\sum_{K=0}^{N} b_K e^{-j\omega K}}{\sum_{K=0}^{N} a_K e^{-j\omega K}} \right\}$$

Example

$$y_n = a y_{n-1} + x_n$$

$$Y(\omega) = a e^{-j\omega} Y(\omega) + X(\omega)$$

$$(1 - a e^{-j\omega}) Y(\omega) = X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1 - a e^{-j\omega}}$$

$$h[n] = a^n u[n]$$

Example

$$y_n - 2r\cos\theta\, y_{n-1} + r^2 y_{n-2} = x_n$$

$$Y(\omega)\left(1 - 2r\cos\theta \quad e^{-j\omega} + r^2 e^{-2j\omega}\right) = X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1 - 2r\cos(\theta)e^{-j\omega} + r^2 e^{-2j\omega}}$$

$$= \frac{1}{(1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{-j\omega})}$$

$$= \frac{A}{1 - re^{j\theta}e^{-j\omega}} + \frac{B}{1 - re^{-j\theta}e^{-j\omega}}$$

solving for $A + B$ yields $(\theta \neq 0)$

$$A = \frac{e^{j\theta}}{2j\sin\theta}$$

$$B = \frac{-e^{-j\theta}}{2j\sin\theta}$$

$$h(n) = \mathcal{F}^{-1}\{H(\omega)\}$$

$$= \frac{e^{j\theta}}{2j\sin\theta}(re^{j\theta})^n u(n) - \frac{e^{-j\theta}}{2j\sin\theta}(re^{-j\theta})^n u(n)$$

$$= \frac{u(n)r^n}{2j\sin\theta}\left[e^{j(n+1)\theta} - e^{-j(n+1)\theta}\right]$$

$$= \underline{\frac{u(n)\, r^n \sin((n+1)\theta)}{\sin(\theta)}}$$

<u>If $\theta = 0$</u>

$$H(\omega) = \frac{1}{(1+re^{-j\omega})^2} \longleftrightarrow h_n = (n+1)(-r)^n u(n)$$