The Discrete Fourier Transform

- The book calls this the Discrete time Fourier Series (DTFS)

- The fast Fourier transform (FFT) is a fast method to implement the DFT.

- Consider the orthogonal basis functions ϕ_K where

$$\langle \phi_K, \phi_\ell \rangle = \underbrace{N}_{\text{integer}} \delta[K-\ell]$$

- ϕ_K are orthogonal, but not normal.

$$\langle \phi_K, \phi_K \rangle = N$$

We can use ϕ_K as the basis of an orthogonal transform

$$x = \sum_{K=0}^{N-1} X_K \phi_K$$

where

$$X_K = \underbrace{\frac{1}{N}}_{\text{factor due to fact that } \langle \phi_K, \phi_K \rangle = N} \langle x, \phi_K \rangle$$

Let the functions ϕ_k be

$$\phi_k[n] = e^{j2\pi \frac{kn}{N}}$$

and let the inner product be

$$\langle f, g \rangle = \sum_{n=0}^{N-1} f[n]\, g^*[n]$$

Claim 1)

$$\langle \phi_k, \phi_\ell \rangle = N\,\delta[k-\ell]$$

for $k \neq \ell$

$$\langle \phi_k, \phi_\ell \rangle = \sum_{n=0}^{N-1} e^{j2\pi \frac{kn}{N}} e^{-j2\pi \frac{\ell n}{N}}$$

$$= \sum_{n=0}^{N-1} e^{j2\pi \frac{(k-\ell)n}{N}}$$

$$= \frac{1 - e^{j2\pi \frac{(k-\ell)}{N} N}}{1 - e^{j2\pi \frac{(k-\ell)}{N}}}$$

$$= \frac{1 - 1}{1 - e^{j2\pi \frac{(k-\ell)}{N}}} = 0$$

for $k=l$

$$\langle \phi_k, \phi_{\cancel{k}} \rangle = \sum_{n=0}^{N-1} e^{j2\pi \frac{kn}{N}} e^{-j2\pi \frac{kn}{N}}$$

$$= \sum_{n=0}^{N-1} 1 = N$$

Using the basis

$$\phi_k[n] = e^{j2\pi \frac{kn}{N}}$$

in the transform

$$\begin{cases} x = \sum_{k=0}^{N-1} X_k \phi_k \\ X_k = \frac{1}{N} \langle x, \phi_k \rangle \end{cases}$$

we get

$$\begin{cases} x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi \frac{kn}{N}} \\ X_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{kn}{N}} \end{cases}$$

DFT

$$X_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{kn}{N}}$$

Inverse DFT

$$x[n] = \sum_{k=0}^{N-1} X_k e^{j2\pi \frac{kn}{N}}$$

Comments

1) X_k is defined for $0 \le k \le N-1$

2) $x[n]$ is defined for $0 \le n \le N-1$

3) But, $x[n]$ is periodic with period N

4) X_k is also periodic with period N.

Example

$$x[n] = 1 \qquad X_k = ?$$

Notice that

$$x[n] = 1 = \phi_0[n] = e^{j2\pi\frac{0n}{N}}$$

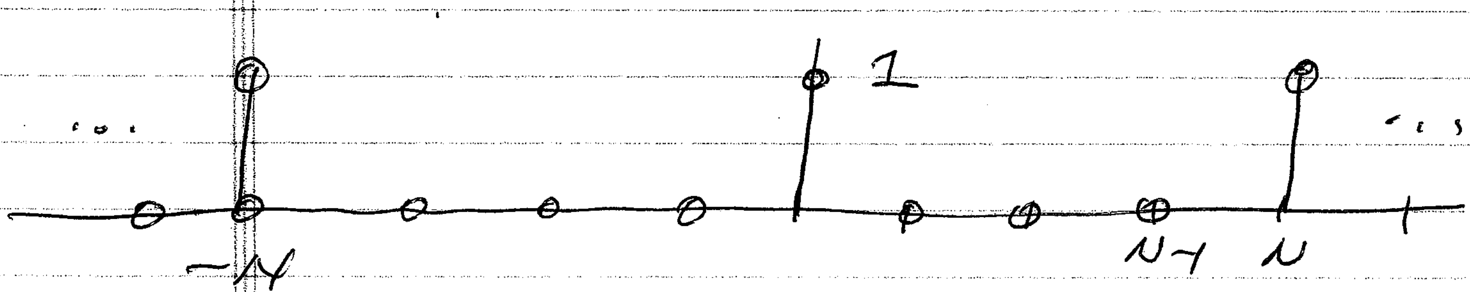
$$X_k = \frac{1}{N}\langle x, \phi_k \rangle$$

$$= \frac{1}{N}\langle \phi_0, \phi_k \rangle = \frac{1}{N} N\delta[k]$$

$$X_k = \begin{cases} 1 & \text{for } k=0 \\ 0 & \text{for } k=1,\dots,N-1 \end{cases}$$

But remember, since X_k is periodic with period N

$$X_k = \sum_{\ell=-\infty}^{\infty} \delta[k - \ell N]$$



$$x[n] = 1 \overset{DFT}{\Longleftrightarrow} X_k = \sum_{\ell=-\infty}^{\infty} \delta[k - \ell N]$$

Example

$$x[n] = e^{j2\pi \frac{mn}{N}} \qquad 0 \le m \le N-1$$

Notice that $x[n] = \phi_m[n]$

So

$$X_k = \frac{1}{N}\langle x, \phi_k \rangle = \frac{1}{N}\langle \phi_m, \phi_k \rangle$$

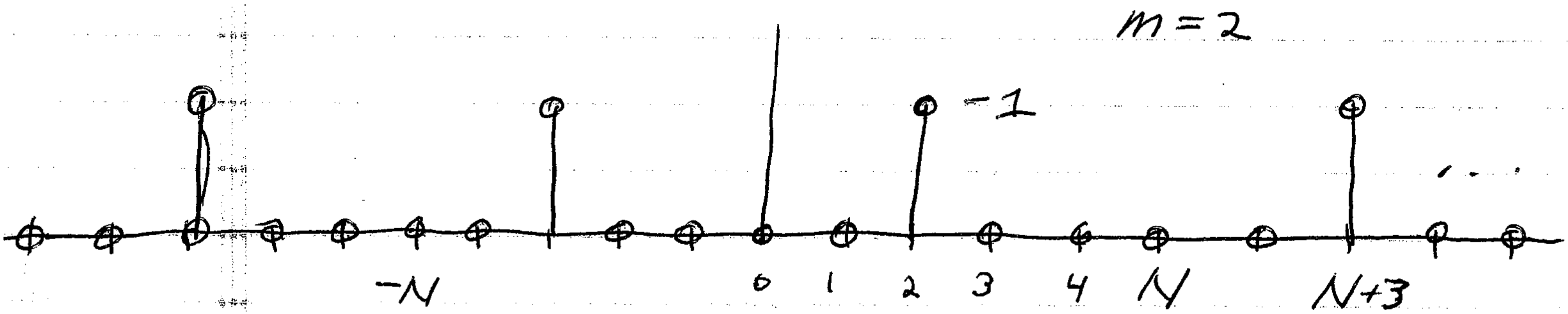
$$= \frac{1}{N} N \delta[k-m]$$

$$X_k = \begin{cases} 0 & \text{for } 0 \le k < m \\ 1 & \text{for } k = m \\ 0 & \text{for } m < k \le N-1 \end{cases}$$

But since X_k is periodic

$$X_k = \sum_{l=-\infty}^{\infty} \delta[k-m-lN]$$

$N=5$
$m=2$



$$x[n] = e^{j2\pi \frac{mn}{N}} \overset{DFT}{\Longleftrightarrow} X_k = \sum_{\ell=-\infty}^{\infty} \delta[k-m-\ell N]$$

Fact: This transform pair holds for m outside the range $0, \ldots, N-1$. Why? Because $e^{j2\pi \frac{mn}{N}}$ is a periodic function of m with period N.

Example:

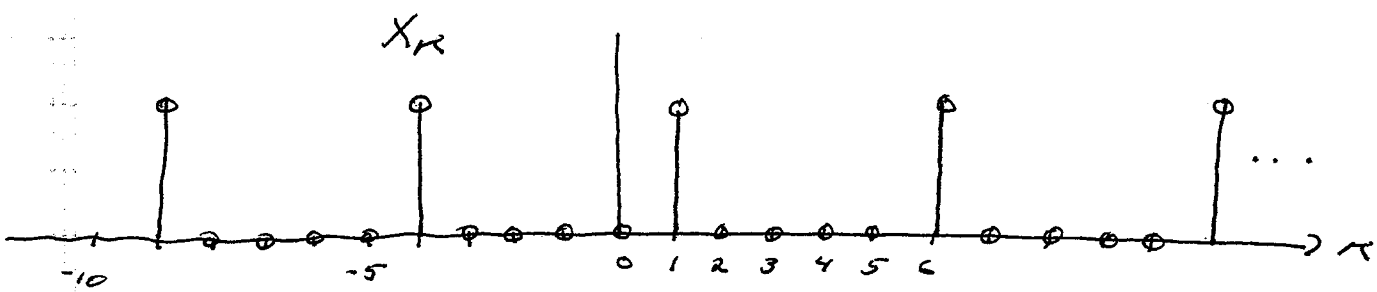
$$x[n] = \cos(2\pi mn/N)$$

$$= \frac{e^{j2\pi mn/N} + e^{-j2\pi mn/N}}{2}$$

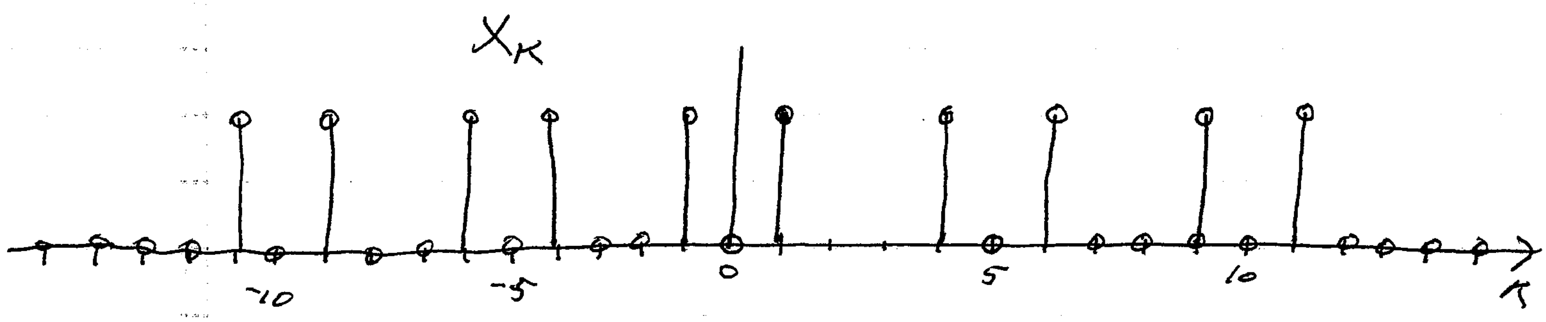
$$X_K = \frac{1}{2}\sum_{\ell=-\infty}^{\infty}\left\{ \delta[K-m-\ell N] + \delta[K+m-\ell N] \right\}$$

<u>$N=5 \quad m=1$</u>

$x[n] = e^{j2\pi \frac{n}{N}} \Leftrightarrow X_K$



$x[n] = \cos(2\pi mn/N) \Leftrightarrow X_K$



Example pulse

$$x[n] = u(n) - u(n-P) \quad \text{for } 0 \le n < N$$

$$X_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j 2\pi \frac{kn}{N}}$$

$$= \frac{1}{N} \sum_{n=0}^{P-1} \overset{1}{\cancel{x[n]}} e^{-j 2\pi \frac{kn}{N}}$$

$$= \frac{1}{N} \frac{1 - e^{-j 2\pi \frac{kP}{N}}}{1 - e^{-j 2\pi \frac{k}{N}}}$$

$$= \frac{1}{N} \frac{e^{-j 2\pi \frac{kP}{2N}}}{e^{-j 2\pi \frac{k}{2N}}} \frac{\left(e^{j 2\pi \frac{kP}{2N}} - e^{-j 2\pi \frac{kP}{2N}}\right)}{\left(e^{j 2\pi \frac{k}{2N}} - e^{-j 2\pi \frac{k}{2N}}\right)}$$

$$= \frac{1}{N} e^{-j 2\pi \frac{k}{N} \frac{(P-1)}{2}} \frac{\sin(2\pi kP/2N)}{\sin(2\pi k/2N)}$$

Definition

$$\text{psinc}_M(x) = \frac{\sin(\pi x M)}{\sin(\pi x)}$$

Properties

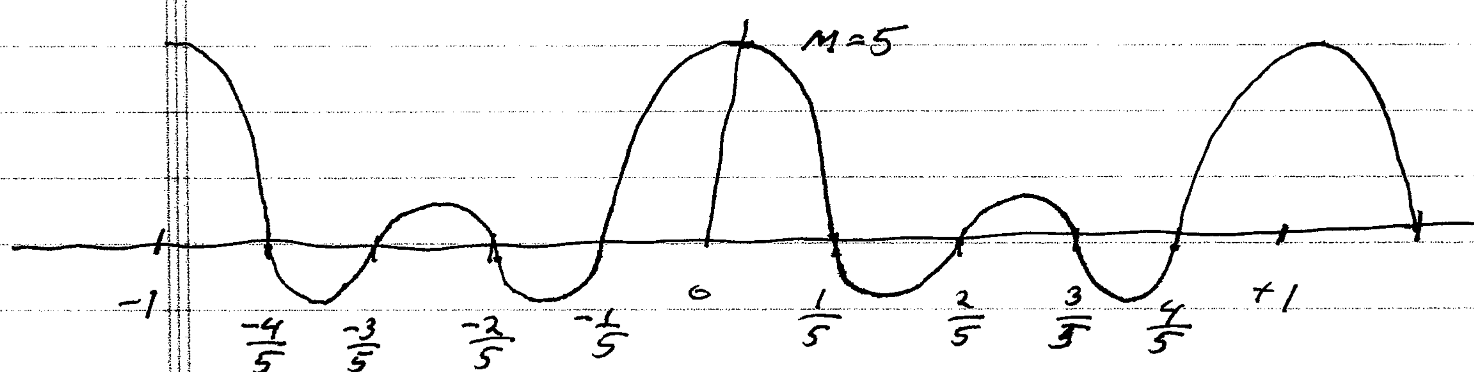
1) $\text{psinc}_M(0) = M$

When M is odd,

2) periodic with period 1

3) Nulls at $x = \frac{l}{M}$ for l an integer

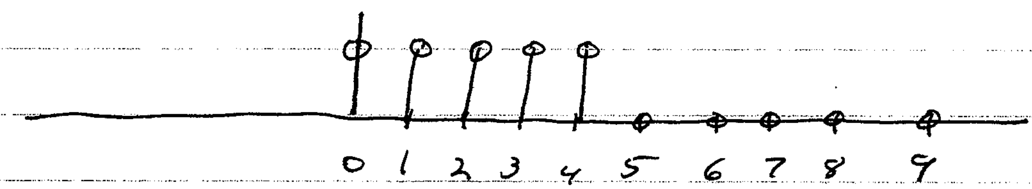
$M=5$



- Like a sinc function, but periodic with period 1.

Consider case when P is odd

$$P = 5, \quad N = 10$$



Center position $= \frac{P-1}{2} = \frac{5-1}{2} = 2$

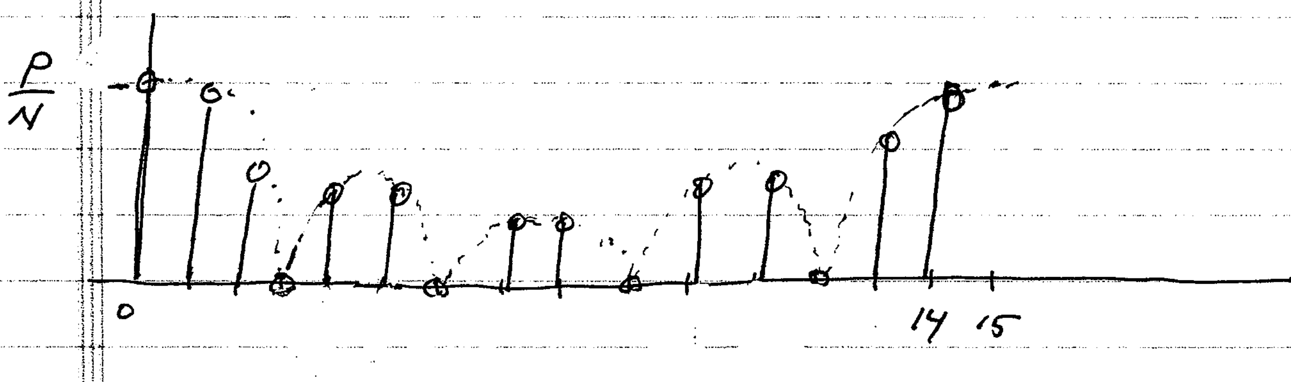
$$x[n] = u(n) - u(n-N)$$

$$\overset{DFT}{\Longleftrightarrow} \quad \underbrace{\frac{1}{N} e^{-j2\pi \frac{k}{N}\left(\frac{P-1}{2}\right)}}_{\text{phase delay of } \frac{P-1}{2}} \quad \underbrace{\frac{\sin(2\pi k P/(2N))}{\sin(2\pi k/(2N))}}_{\text{psinc}_P(k/N)}$$

$$X_k = \frac{1}{N} e^{-j2\pi \frac{k}{N}\left(\frac{P-1}{2}\right)} \text{psinc}_P(k/N)$$

$$|X_k| = \frac{1}{N} \left| \text{psinc}_P(k/N) \right|$$

$|X_k|$ $\quad p = 5 \quad N = 15$



DFT PAIRS

$$x[n] = 1 \overset{DFT}{\Longleftrightarrow} X_K = \sum_{\ell=-\infty}^{\infty} \delta[K - \ell N]$$

$$x[n] = e^{j2\pi \frac{mn}{N}} \overset{DFT}{\Longleftrightarrow} X_K = \sum_{\ell=-\infty}^{\infty} \delta[K - m - \ell N]$$

$$x[n] = \cos(2\pi mn/N) \overset{DFT}{\Longleftrightarrow} X_K = \frac{1}{2} \sum_{\ell=-\infty}^{\infty} \Big\{ \delta[K-m-\ell N] + \delta[K+m-\ell N] \Big\}$$

$$x[n] = \sin(2\pi mn/N) \overset{DFT}{\Longleftrightarrow} X_K = \frac{1}{2j} \sum_{\ell=-\infty}^{\infty} \Big\{ \delta[K-m-\ell N] - \delta[K+m-\ell N] \Big\}$$

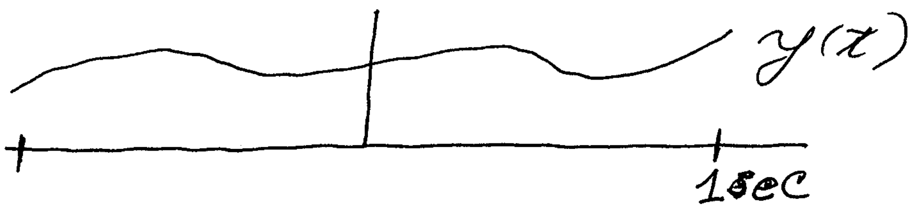
$$x[n] = u(n) - u(n-p) \overset{DFT}{\Longleftrightarrow}$$

$$X_K = \frac{1}{N} e^{-j2\pi \frac{K}{N}\left(\frac{p-1}{2}\right)} \, p \, \mathrm{sinc}_p(K/N)$$

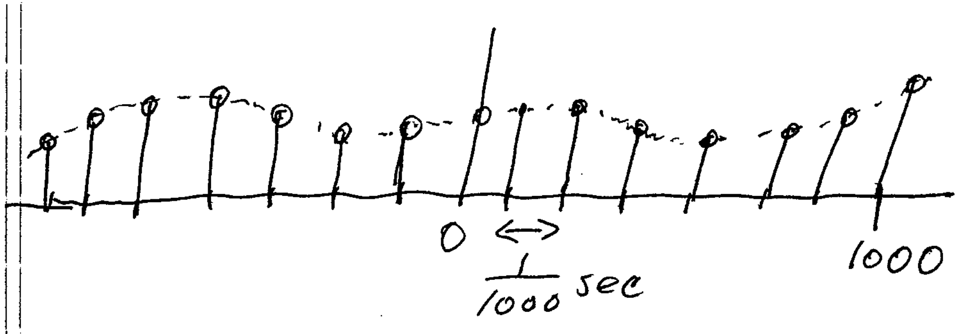
Example

$$y(t) = \sin(2\pi 50 t) + 0.5 \sin(2\pi 20 t)$$

Sampling Rate 1000 Hz (samples/sec)



$$x_n = y(Tt) \quad T = \frac{1}{1000}\text{ sec}$$



We observe the signal from $0 \le t < 1\text{sec}$

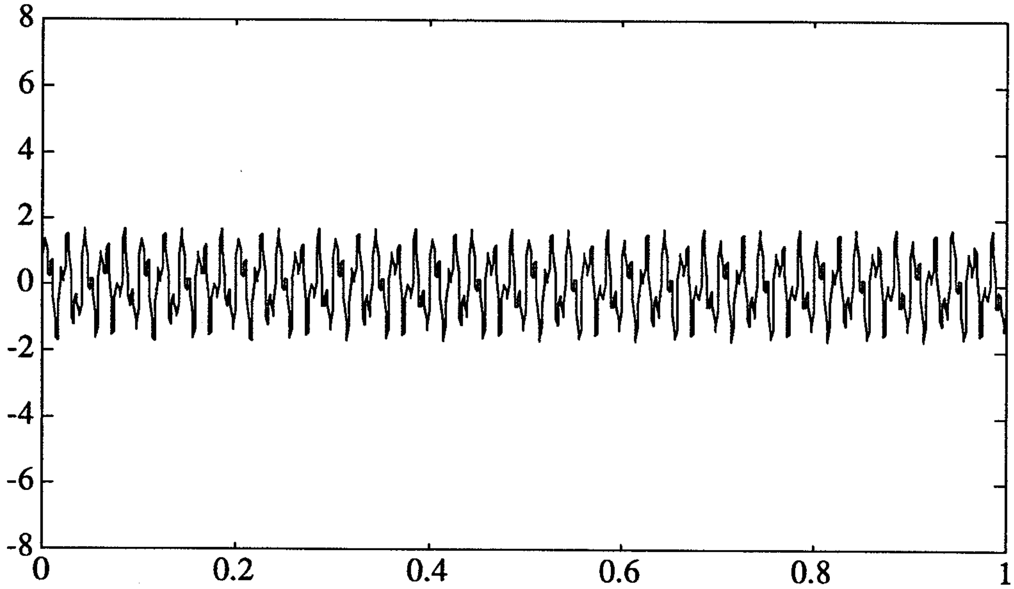
$$\Rightarrow \quad 0 \le n \le \frac{1\text{sec}}{T\text{sec}} = 1000 = N$$

$$\boxed{x_n = \sin\left(2\pi \frac{50}{1000} n\right) + \frac{1}{2}\sin\left(2\pi \frac{20}{1000} n\right) \qquad 0 \le n \le N-1}$$

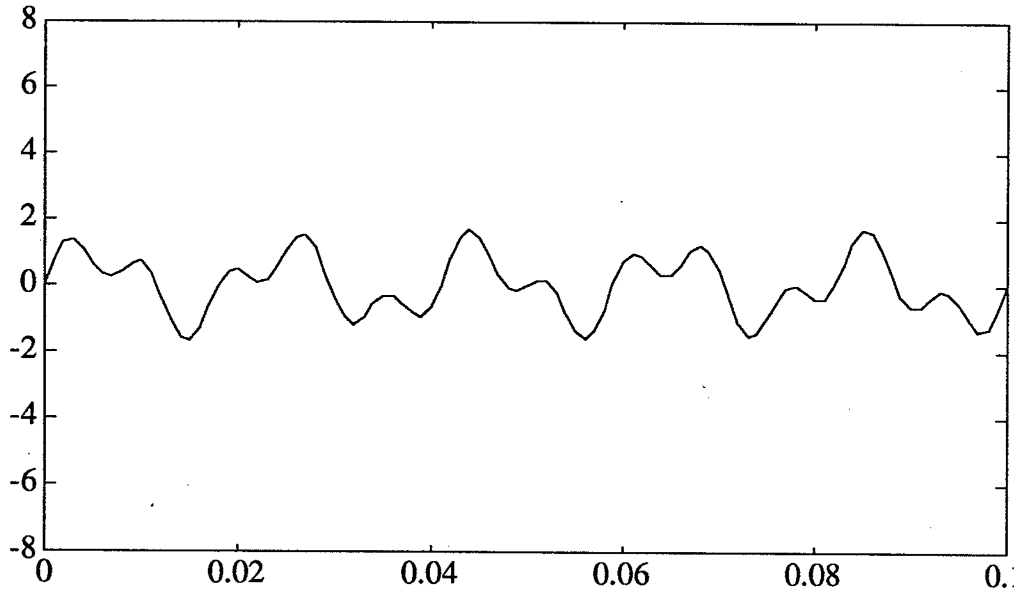
DFT X_n = is

$$a_K = \begin{cases} \frac{1}{2j} & \text{if } K = 50 \\ -\frac{1}{2j} & \text{if } K = 1000-50 \\ \frac{1}{4j} & \text{if } K = 20 \\ \frac{-1}{4j} & \text{if } K = 1000-20 \end{cases}$$

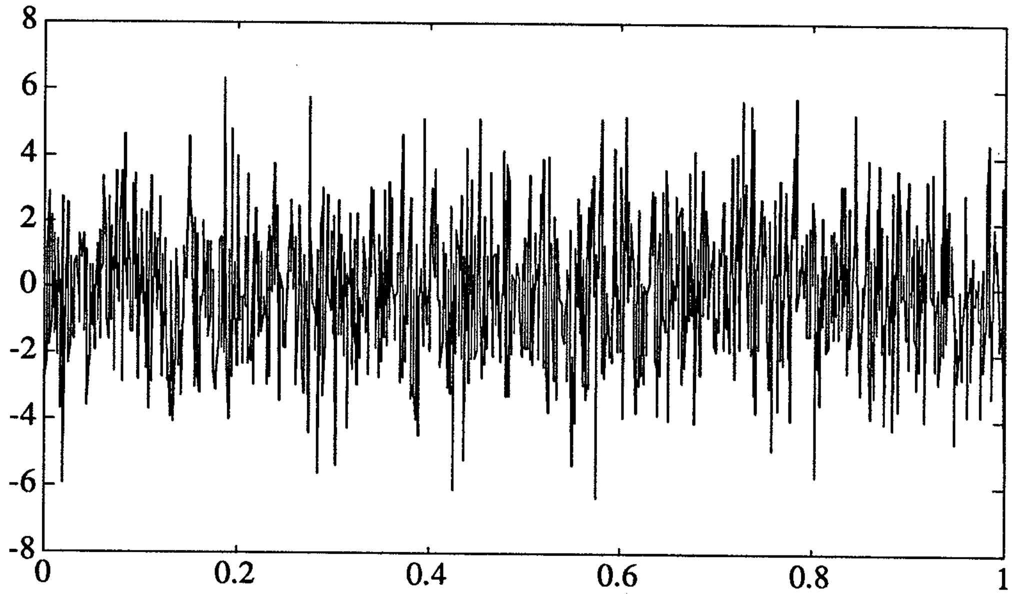
SIGNAL: $y(t) = \sin(2\pi\, 50\, t) + 0.5 \sin(2\pi\, 120\, t)$

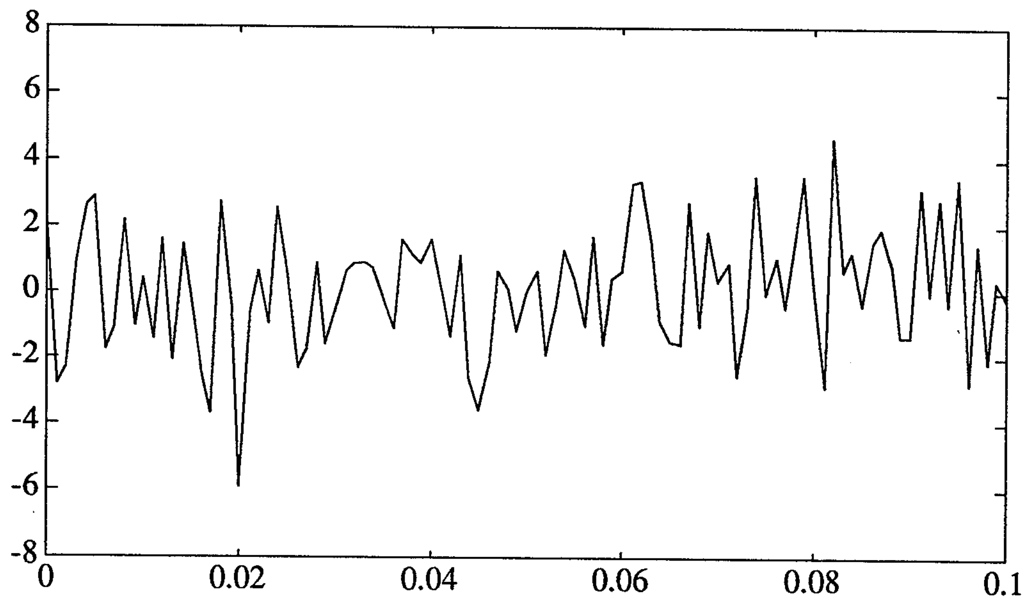


Sampling Rate = 1000 Hz

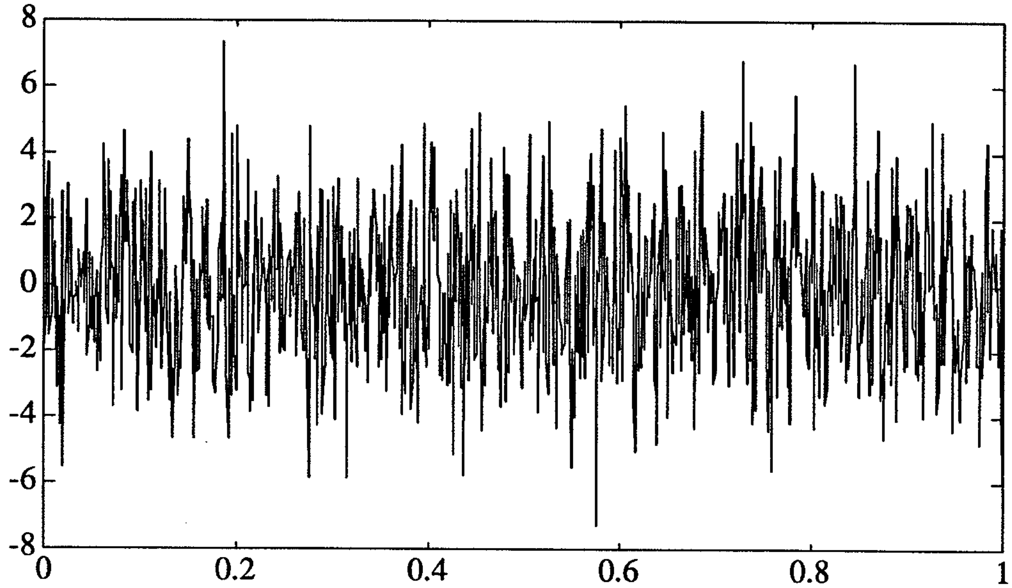


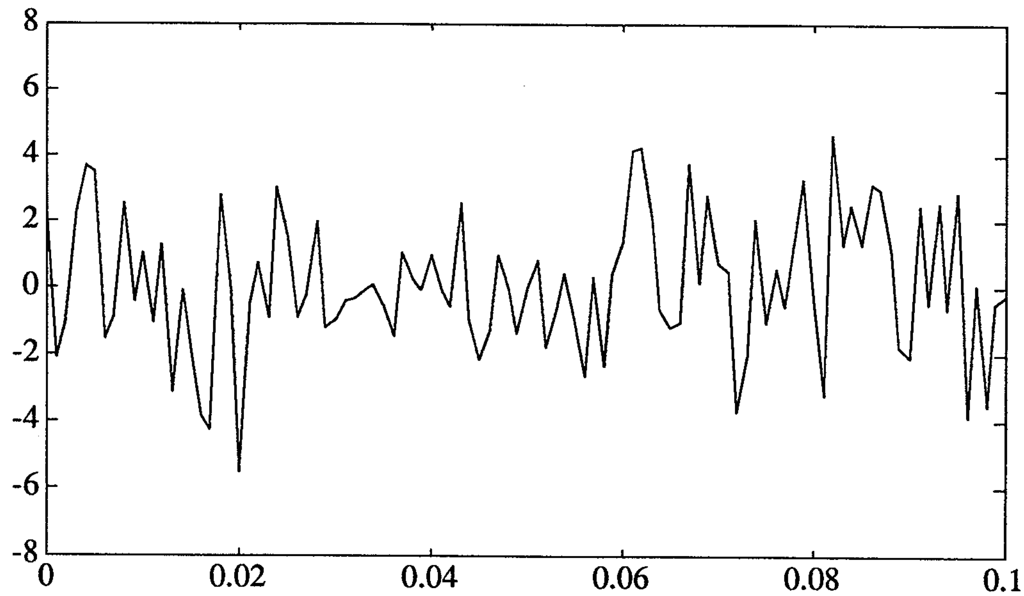
NOISE: i.i.d. zero-mean unit-variance Gaussians

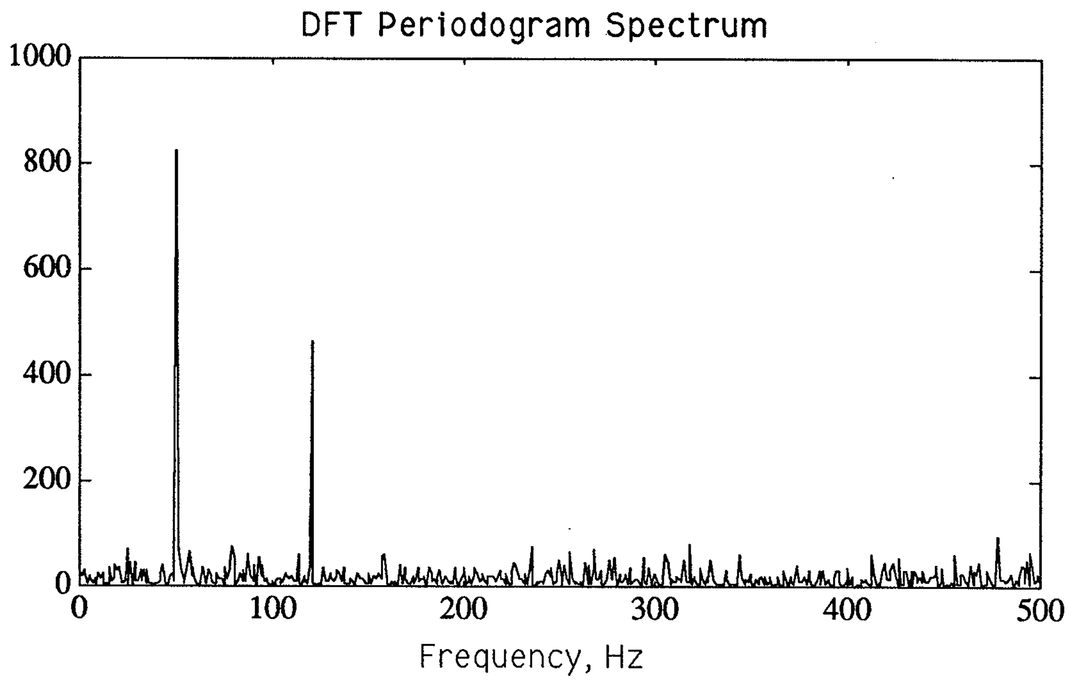




SIGNAL + NOISE






DFT Periodogram Spectrum
1000
800
600
400
200
0
0
100
200
300
400
500
Frequency, Hz

DFT properties

Time domain convolution $\Longleftrightarrow$ Frequency domain multiplication

If x_n and y_n are periodic DT signals

$$\sum_{k=-\infty}^{\infty} x_k \, y_{n-k} = \infty$$

Instead we use periodic convolution

$$x_n \circledast y_n = \sum_{k=0}^{N-1} x_k \, y_{n-k} = z_n$$

$$Z_k = X_k \, Y_k$$

1. Periodic Convolution

$$x_n \circledast y_n \Longleftrightarrow N X_k \, Y_k$$

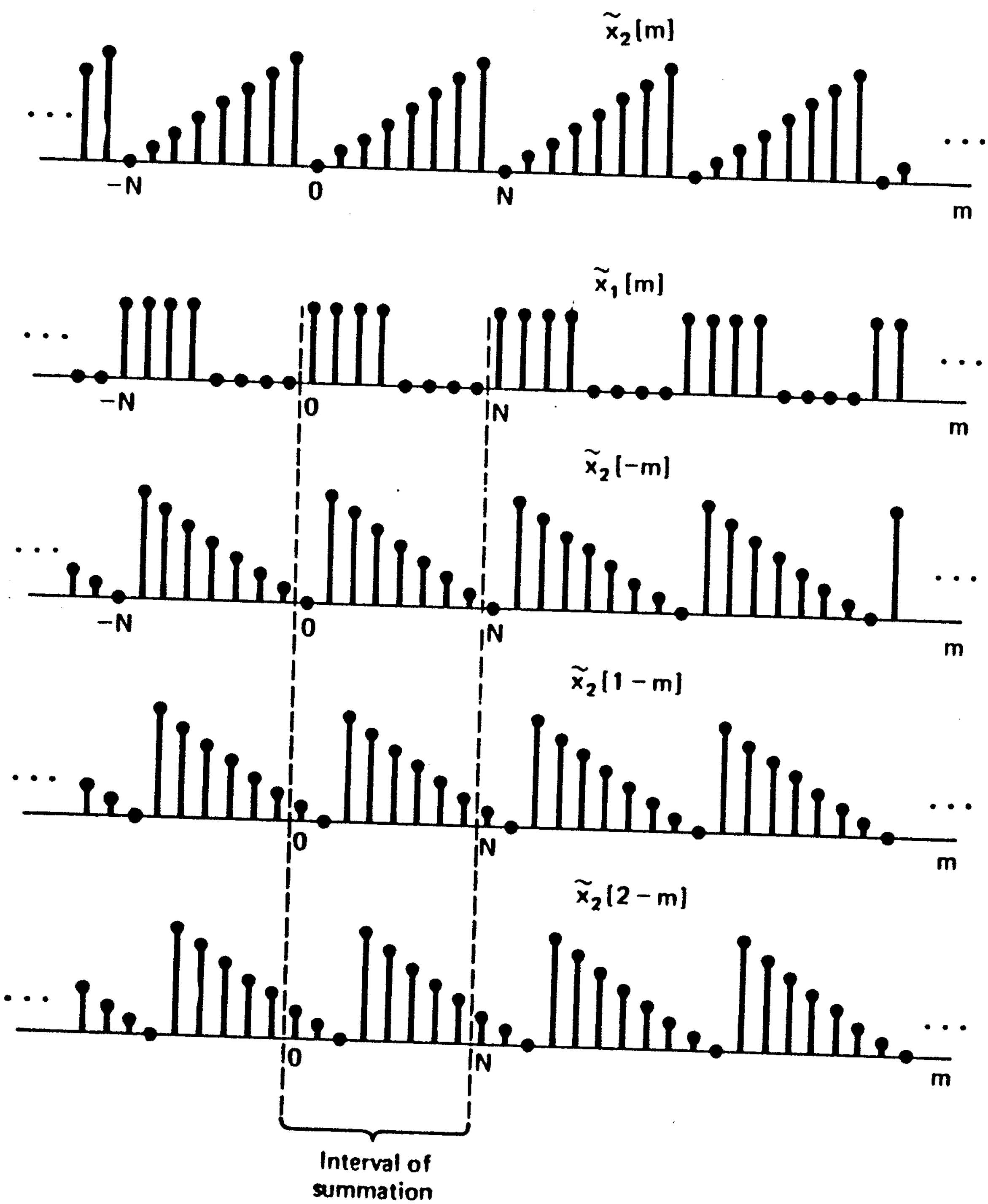


Figure 5.22 Procedure in forming the periodic convolution of two periodic sequences.

Periodic Convolution

1) $\tilde{x}_n$ and $\tilde{y}_n$ are periodic

$$\tilde{x}_n \circledast \tilde{y}_n = \sum_{k=0}^{N-1} \tilde{x}_k \tilde{y}_{n-k}$$

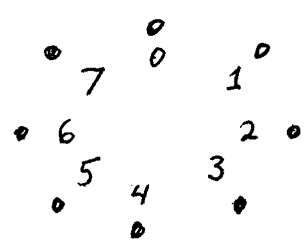
2) x_n and y_n defined for $n = 0, \ldots, N-1$

We need a new operation

$$n \bmod N = \begin{cases} \text{\# between } 0 \text{ and } N-1 \\ \text{such that} \\ \quad n = kN + n \bmod N \\ \text{for } k \text{ an integer} \end{cases}$$

modulo

Interpretation for $N=8$



to calculate $n \bmod 8$ move n positions clockwise

$1 \bmod 8 = 1$

$8 \bmod 8 = 0$

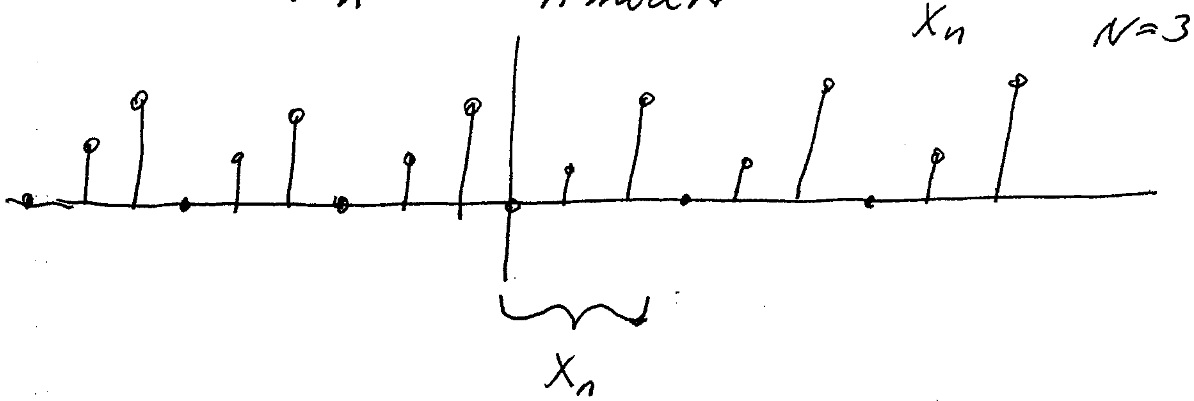
$9 \bmod 8 = 1$

$17 \bmod 8 = 1$

$-1 \bmod 8 = 7$

The periodic version of x_n is given by

$$\tilde{x}_n = x_{n \bmod N}$$



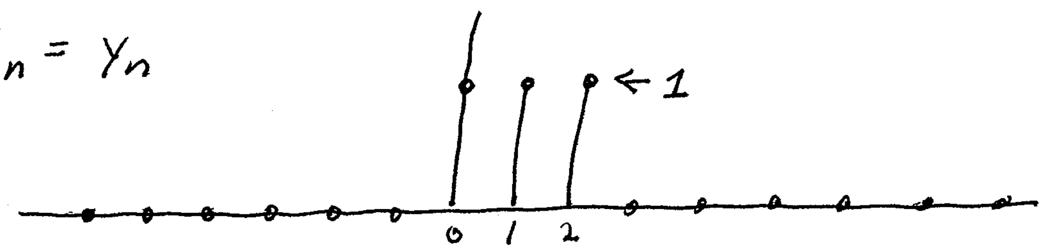
$$x_n \circledast y_n = \sum_{k=0}^{N-1} \tilde{x}_k \, \tilde{y}_{n-k}$$

$$\boxed{x_n \circledast y_n = \sum_{k=0}^{N-1} x_k \, y_{(n-k) \bmod N}}$$

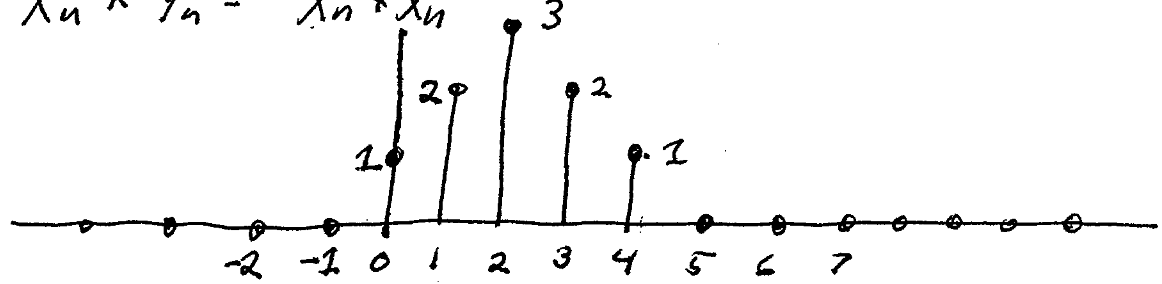
Example

We would like to compute the standard convolution of X_n and Y_n using the DFT.

$X_n = Y_n$

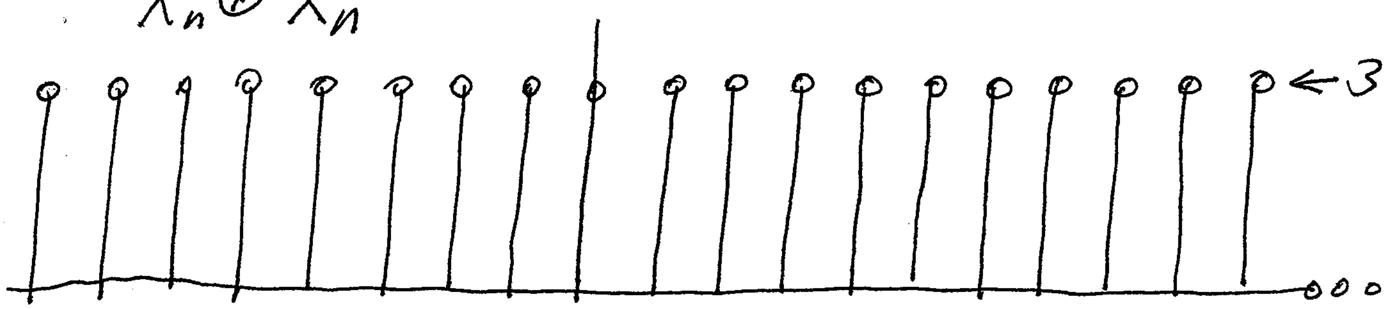


$X_n * Y_n = X_n * X_n$



We can use the DFT to compute the cyclic convolution for $N=3$

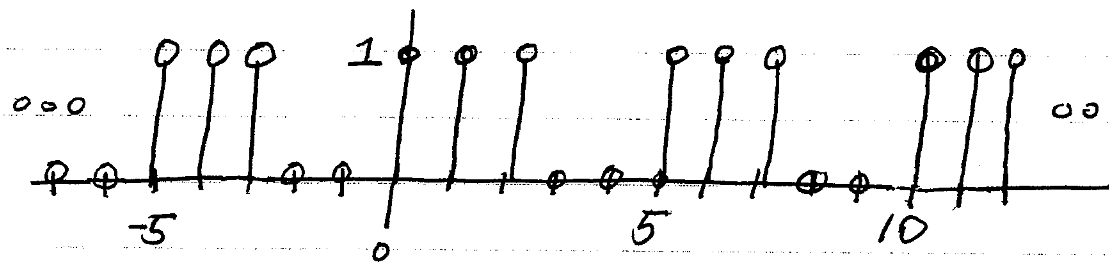
$X_n \circledast X_n$



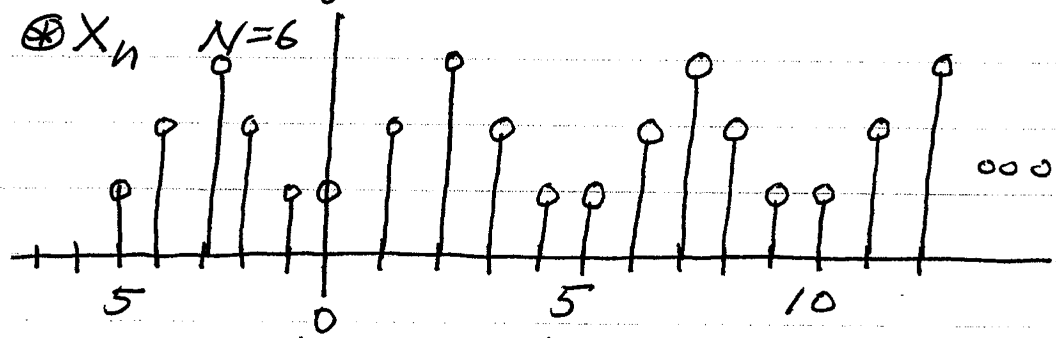
Why because $X_{n \bmod 3} = 1$ for all n!

Answer: Pick $N \geq 2 \cdot 3 - 1 = 5$

$\tilde{X}_n = X_{n \bmod}$



$Z_n = X_n \circledast X_n \quad N=6$



Result of DFT are in samples $n = 0, \ldots, \underbrace{4}_{N-1}$

$$X_n * X_n = \begin{cases} Z_n & \text{for } 0 \leq n < 5 \\ 0 & \text{otherwise} \end{cases}$$

↑ normal convolution

Computation

If X_n has length M

Direct computation requires

$(2M+1)^2$ operations (multiplies)

DFT computation requires

$2\,(2M\ 1 \text{ point DFT})$

$+\ 2M-1$ multiplies

$+1\,(2M-1 \text{ point } DFT^{-1})$

(FFT)

$= \quad 2\,(2M-1)\log(2M\ 1)$ (2 forward DFT's)

$+(2M-1)$ (weights)

$+(2M-1)\log(2M-1)$ (1 inverse DFT)

$= \quad 3(2M-1)(\log(2M-1)+1)$

much less for large M!

$\boxed{\text{order } M\log M}$

2. Duality

$$DFT\{g_n\} = f_k$$

$$DFT\{f_n\} = \frac{1}{N} g_{-k}$$

+ A Bunch more properties

Just remember - Treat x_n as one period of a periodic signal.