Review of Concepts from Formal Logic

1 Basic Logical Operations

In order to prove statements or theorems, it is necessary to understand some basic principals of logic. The basic logical operators are listed below, along with their definitions. In each case, the symbols P or Q represent logical variables or statements that are either true or false. The logical operators are then defined in terms of a truth table.

- **Not:** The symbol $\neg$ is the logical negation operator. So its truth table is given by

P	$\neg P$
True	False
False	True

- **Or:** The symbol $\vee$ is the logical "or" operator. So its truth table is given by

P	Q	$P \vee Q$
True	True	True
True	False	True
False	True	True
False	False	False

- **And:** The symbol & is the logical "and" operator. So its truth table is given by

P	Q	$P \& Q$
True	True	True
True	False	False
False	True	False
False	False	False

- **If Then:** The symbol $\Rightarrow$ is the logical implication operator. The statement "if P then Q" may be written as $P \Rightarrow Q$, and has the truth table is given by

P	Q	$P \Rightarrow Q$
True	True	True
True	False	False
False	True	True
False	False	True

- **If And Only If:** The symbol $\Leftrightarrow$ is the bi-directional implication operator. The statement "P if and only if Q" or "P iff Q" may be written as $P \Leftrightarrow Q$, and has the truth table is given by

P	Q	$P \Leftrightarrow Q$
True	True	True
True	False	False
False	True	False
False	False	True

Consider the logical statement $P \Rightarrow Q$. Then the following three statements are often associated with it.

- Converse $Q \Rightarrow P$
- Inverse $\neg P \Rightarrow \neg Q$
- Contrapositive $\neg Q \Rightarrow \neg P$

We will illustrate these three statements with the example logical statement, "If an animal is a cat then it has a tail". This is a true statment for a normal cat.

Converse: The converse of a statement is not generally true. The converse of the example statment is "If an animal has a tail then it is a cat." This statement is generally not true because there are other animals, such as dogs, that have tails but are not cats. However, if both the statement $P \Rightarrow Q$ and its converse are true, then we know that $P \Leftrightarrow Q$. In this case, we sometimes say that P is true if and only if Q is true.

Inverse: The inverse of a statement is not generally true. The inverse of the example statement is "If an animal is not a cat then it does not have a tail." This statement is generally not true for the same reason that the converse is not always true. Once again, dogs are not cats, but they do have tails.

Contrapositive: The contrapositive of a statement is always true when the orginal statement is true. (i.e., It has the same true value as the original statement.) For the example statement, the contrapositive is "If an animal does not have a tail, then it can not be a cat." The contrapositive is often a useful way of looking at a logical statement.

2 Negation of Logical Operators

It is often the case, that we need to negate a logical statement. This happens if we need to show that a particular property is false. For example, consider the property that, "If Bob is a Purdue student then he is happy." We can represent this as a logical expression by first defining the two properties

$$\begin{aligned} P &= \text{Bob is a Purdue student} \\ Q &= \text{Bob is happy} \ , \end{aligned}$$

and then forming the logical expression.

$$Prop1 \Leftrightarrow (P \Rightarrow Q) \ .$$

However, after spending considerable effort to prove $Prop1$, we might begin to suspect that this property is false. Of course, just because we can not prove that something is true, that does not mean that it is false. In order to **prove** that property $Prop1$ is false, we must prove the negation of the property.

$$\neg Prop1 \Leftrightarrow \neg (P \Rightarrow Q) \ .$$

In fact, we can use a truth table to show that the negation of $P \Rightarrow Q$ is given by $P \& \neg Q$. So in order to show that $Prop1$ is false, we must show that

$$\neg Prop1 \Leftrightarrow P \& \neg Q \ ,$$

where

$$\begin{aligned} P &= \text{Bob is a Purdue student} \\ \neg Q &= \text{Bob is not happy} \ . \end{aligned}$$

So in words, we must show that "Bob is a Purdue student, and he is not happy."

Below we list some commonly used negations of logical expressions.

$$\begin{aligned} \neg (P \Rightarrow Q) &\Leftrightarrow P \& \neg Q \\ \neg (P \& Q) &\Leftrightarrow \neg P \vee \neg Q \\ \neg (P \vee Q) &\Leftrightarrow \neg P \& \neg Q \\ \neg (P \Leftrightarrow Q) &\Leftrightarrow (\neg P \& Q) \vee (P \& \neg Q) \end{aligned}$$

3 The "For All" and "Exists" Qualifiers

In many cases, mathematical statements are made using the qualifiers "for all" and "there exists". These qualifiers state properties about elements in a set. For example, we might ask the question whether all elements x in the set S have the property Px.

- **For All:** The symbol $\forall$ is the "for all" qualifier. The statement "for all $x \in S$, x has property P" may be written as $\forall x \in S, Px$.
- **There Exists:** The symbol $\exists$ is the "there exists" qualifier. The statement "there exist an $x \in S$ such that x has property P" may be written as $\exists x \in S \, s.t. Px$.

It is often the case that a statement involving a $\forall$ or $\exists$ qualifier must be negated. The rule for negating quantifiers is easy. Simply swap the $\forall$ and $\exists$ symbols, and negate the associated statements. So for example, negation of the $\forall$ quantifier results in the following.

$$\neg (\forall x \in S, Px) \quad \Leftrightarrow \quad \exists x \in S \, s.t. \neg Px$$

Intuitively, if it is not true that all x have property P, then it must be true that at least one x has property $\neg P$. Similarly, negation of the $\exists$ quantifier results in

$$\neg (\exists x \in S \, s.t. \, Px) \quad \Leftrightarrow \quad \forall x \in S, \neg Px \ .$$

In some cases, the quantifiers may be used together. This case can be handled by simply applying the rules repeatedly. In fact, the same rules can also be used when quantifiers are nested.

$$\begin{aligned} \neg (\forall x \in A, \exists y \in B, \, s.t. \, Pxy) &\Leftrightarrow \exists x \in A, \, s.t. \neg (\exists y \in B, \, s.t. \, Pxy) \\ &\Leftrightarrow \exists x \in A, \, s.t. \forall y \in B, \neg Pxy \end{aligned}$$

In certain cases, the property in question may itself be a logical expression. So consider the statement

$$\neg (\forall x \in S, Px \Rightarrow Qx) \ .$$

This means that

$$\exists x \in S \, s.t. \neg \left(Px \Rightarrow Qx \right) \ .$$

But it is easily varified that $\neg(Px \Rightarrow Qx)$ is equivalent to $\neg Qx \,\&\, Px$; so we know that

$$\neg \left(\forall x \in S, Px \Rightarrow Qx \right) \Leftrightarrow \exists x \in S \, s.t. \left(\neg Qx \,\&\, Px \right) \ .$$

4 Some examples

Nested quantifiers occur commonly in practical situations. For example, let us say that a university is "good" if "all students have a class that they like." This statement may be stated formally by defining A to be the set of all students, B to be the set of all classes, and Pxy to be the property that student x likes class y. So then Purdue is a "good" university if the following statement is true.

$$\forall x \in A, \exists y \in B, \ s.t. \, Pxy$$

However, in order to show that Purdue is **not** a good university, we must show that

$$\neg \left(\forall x \in A, \exists y \in B, \ s.t. \, Pxy \right)$$

which is equivalent to the statement that

$$\exists x \in A, \ s.t. \, \forall y \in B, \neg Pxy \ .$$

So in order to prove that Purdue is not a good university, we need to find a student who dislikes all of his/her classes.