ECE 301 HW9 Solutions

1) $X_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{j2\pi \frac{kn}{N}}$

a: $x[n] = \delta[n]$ for $0 \le n < N$

$X_k = \frac{1}{N}\sum_{n=0}^{N-1} \delta[n]\, e^{j2\pi \frac{kn}{N}} = 1/N$

b: $x[n] = \delta[n-m]$

$X_k = \frac{1}{N}\sum_{n=0}^{N-1} \delta(n-m)\, e^{j2\pi \frac{kn}{N}} = \frac{1}{N} e^{j\frac{2\pi km}{N}}$

c: $x[n] = e^{j\frac{2\pi nm}{N}}$ $\quad m < N$

$x[n] = \phi_m[n] \Rightarrow X_k = \delta[k-m]$

d: $x[n] = \cos\left(\frac{2\pi nm}{N}\right)$

$x[n] = \frac{1}{2}\left(e^{j2\pi\frac{nm}{N}} + e^{-j\frac{2\pi nm}{N}}\right) \Rightarrow X_k = \frac{1}{2}\delta[k-m] + \frac{1}{2}\delta[k-(N-m)]$

e: $x[n] = \sin\left(\frac{2\pi nm}{N}\right)$

$x[n] = \frac{1}{2j}\left(e^{j\frac{2\pi nm}{N}} - e^{-j\frac{2\pi nm}{N}}\right) \Rightarrow X_k = \frac{1}{2j}\delta[k-m] - \frac{1}{2j}\delta[k-(N-m)]$

2) $x[n] = e^{j\omega n} \qquad 0 \le n < N$

a: $X_k = \frac{1}{N} \sum_{n=0}^{N-1} e^{j\omega n} e^{-j\frac{2\pi k n}{N}}$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \left(e^{j(\omega - \frac{2\pi k}{N})}\right)^n = \frac{1}{N} \frac{1 - \left(e^{j(\omega - \frac{2\pi k}{N})}\right)^N}{1 - e^{j(\omega - 2\pi k/N)}}$$

$$= \frac{1}{N} \frac{1 - e^{j(\omega N - 2\pi k)}}{1 - e^{j(\omega - 2\pi k/N)}} =$$

$$= \frac{1}{N} \frac{1 - e^{j\omega N}}{1 - e^{j(\omega - 2\pi k/N)}} \qquad \text{since } e^{-j2\pi k} = 1 \text{ for } k \in \mathbb{Z}$$

$$= \frac{1}{N} \frac{e^{j\frac{\omega N}{2}}}{e^{j(\omega/2 - \pi k/N)}} \frac{e^{-j\omega N/2} - e^{j\omega N/2}}{e^{-j(\omega/2 - \pi k/N)} - e^{+j(\omega/2 - \pi k/N)}}$$

$$= \frac{1}{N} e^{j(\frac{\omega N}{2} - \frac{\omega}{2} + \frac{\pi k}{N})} \frac{-2j \sin(\frac{\omega N}{2})}{-2j \sin(\frac{\omega}{2} - \frac{\pi k}{N})}$$

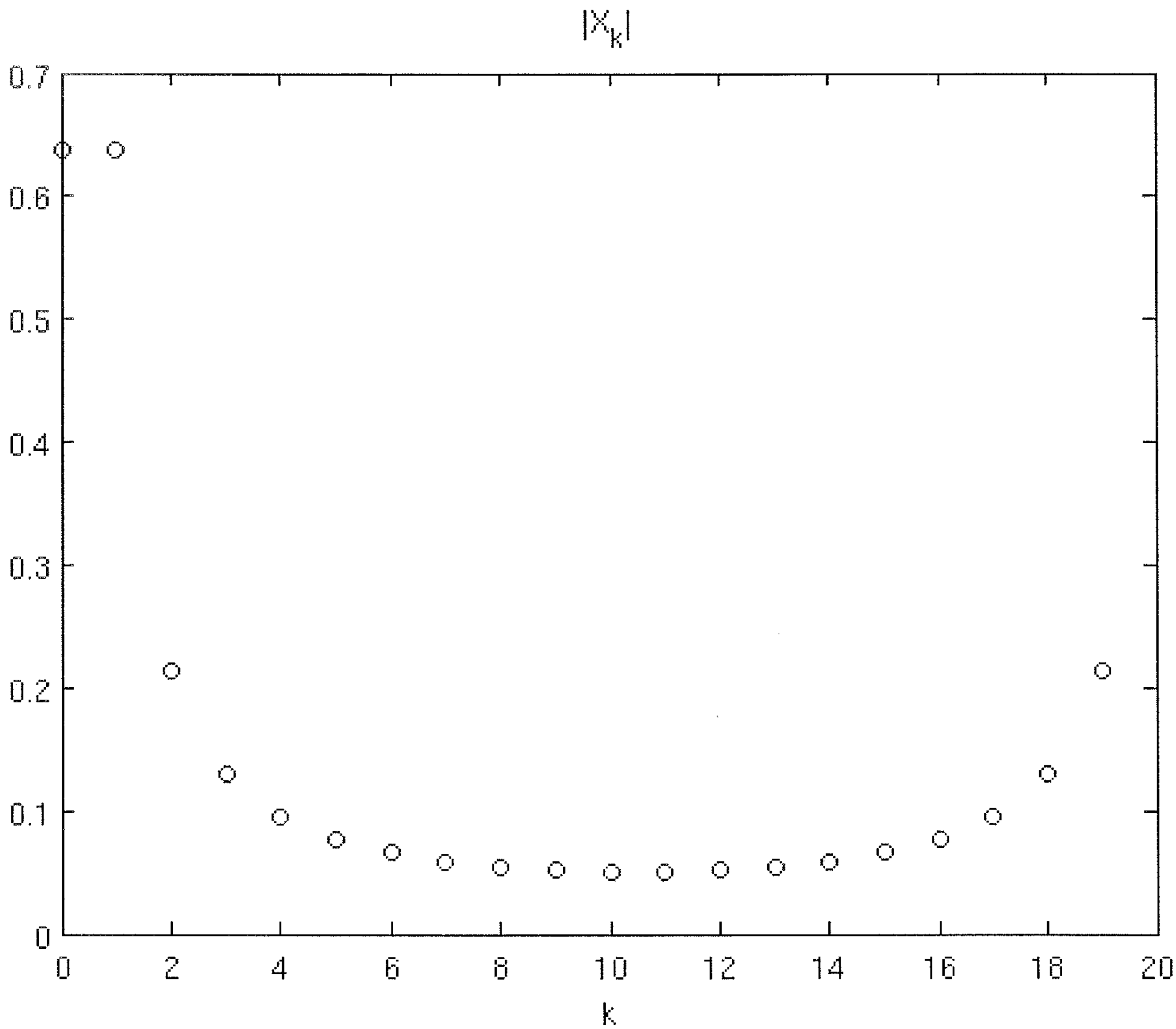
$$= \frac{1}{N} e^{j(\frac{\omega}{2}(N-1) + \frac{\pi k}{N})} \frac{\sin(\frac{\omega N}{2})}{\sin(\frac{\omega}{2} - \frac{\pi k}{N})}$$

b: $\omega = \frac{\pi}{N} \qquad N = 20$

$$X_k = \frac{1}{20} e^{j(\frac{\pi}{40}(19) + \frac{\pi k}{20})} \frac{\sin(\frac{\pi}{2})}{\sin(\frac{\pi}{40} - \frac{\pi k}{20})}$$

$$|X_k| = \frac{1}{20} \left| \frac{1}{\sin(\frac{\pi}{40} - \frac{\pi k}{20})} \right|$$

See plot for sketch



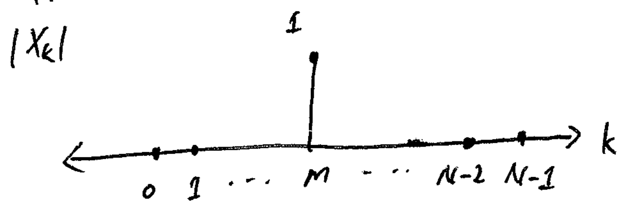
$|X_k|$
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 2 4 6 8 10 12 14 16 18 20
k

c: $\omega = \frac{2\pi m}{N} \quad m \in \mathbb{Z}$

$$x[n] = e^{j\frac{2\pi m n}{N}}$$

$$= \sum_{k=0}^{N-1} X_k \, e^{j\frac{2\pi k n}{N}} \quad \Rightarrow \quad X_k = \delta[k-m]$$

d:



3) a: $\langle \phi_k, \phi_\ell \rangle = \alpha \delta[k-\ell]$

$$x[n] = \sum_{k=0}^{N-1} X_k \phi_k[n]$$

$$|x[n]|^2 = x[n]\, x^*[n] = \sum_{k=0}^{N-1} X_k \phi_k[n] \sum_{\ell=0}^{N-1} X_\ell^* \phi_\ell^*[n]$$

So, $$\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} X_k X_\ell^* \phi_k[n] \phi_\ell^*[n]$$

$$= \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} X_k X_\ell^* \underbrace{\sum_{n=0}^{N-1} \phi_k[n] \phi_\ell^*[n]}_{\langle \phi_k[n], \phi_\ell[n] \rangle}$$

$$= \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1} X_k X_\ell^* \alpha \delta[k-\ell]$$

$$= \alpha \sum_{k=0}^{N-1} |X_k|^2$$

b: $\phi_k[n] = e^{j \frac{2\pi k n}{N}}$ $\qquad \alpha = N$

4) $X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$ $\qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$

a: $x[n] = u[n] - u[n-m] \qquad m \geq 0$

$$X(\omega) = \sum_{n=0}^{m-1} (1) e^{-j\omega n} = \frac{1-e^{-j\omega m}}{1-e^{-j\omega}} = \frac{e^{-j\frac{\omega m}{2}}}{e^{-j\frac{\omega}{2}}} \frac{e^{+j\frac{\omega m}{2}} - e^{-j\frac{\omega m}{2}}}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}$$

$$= e^{-j\frac{\omega}{2}(m-1)} \frac{\sin(\frac{\omega m}{2})}{\sin(\frac{\omega}{2})}$$

b: $x[n] = \delta[n-m] \qquad m \in \mathbb{Z}$

$$X(\omega) = \sum_{n=-\infty}^{\infty} \delta[n-m] e^{-j\omega n} = e^{-j\omega m}$$

c: $x[n] = e^{(j\omega_0 - a)n} u[n]$

$$X(\omega) = \sum_{n=0}^{\infty} e^{(j\omega_0 - j\omega - a)n} = \frac{1}{1-e^{-a}e^{j(\omega_0-\omega)}}$$

d: $x[n] = \cos(\omega_0 n + \phi) = \frac{1}{2}e^{j\phi}e^{j\omega_0 n} + \frac{1}{2}e^{-j\phi}e^{-j\omega_0 n}$

$$X(\omega) = \frac{1}{2}e^{j\phi}\delta(\omega-\omega_0) + \frac{1}{2}e^{-j\phi}\delta(\omega+\omega_0)$$

e: $x[n] = \sin(\omega_0 n + \phi) = \frac{1}{2j}e^{j\phi}e^{j\omega_0 n} - \frac{1}{2j}e^{-j\phi}e^{-j\omega_0 n}$

$$X(\omega) = \frac{1}{2j}e^{j\phi}\delta(\omega-\omega_0) - \frac{1}{2j}e^{-j\phi}\delta(\omega+\omega_0)$$

f: $x[n] = a^n u[n] \qquad |a| < 1$

$$X(\omega) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1-ae^{-j\omega}}$$

g: $x[n] = a^{|n|} \qquad |a| < 1$

$$X(\omega) = \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{n=-\infty}^{0} a^{-n} e^{j\omega n} - a^{(0)}$$

$$= \sum_{n=0}^{\infty} \left(a e^{-j\omega}\right)^n + \sum_{n=0}^{\infty} \left(a e^{j\omega}\right)^n - 1$$

$$= \frac{1}{1 - a e^{-j\omega}} + \frac{1}{1 - a e^{j\omega}} - 1 = \frac{1 - a e^{j\omega} + 1 - a e^{-j\omega}}{1 - a e^{j\omega} - a e^{-j\omega} + a^2} - 1$$

$$= \frac{2 - 2a\cos(\omega)}{1 + a^2 - 2a\cos(\omega)} - 1 = \frac{2 - 2a\cos(\omega) - 1 - a^2 + 2a\cos(\omega)}{1 + a^2 - 2a\cos(\omega)}$$

$$= \frac{1 - a^2}{1 - 2a\cos(\omega) + a^2}$$

h: $x[n] = n a^n u[n] \qquad |a| < 1$

$$X(\omega) = \sum_{n=0}^{\infty} n a^n e^{-j\omega n} = \sum_{n=0}^{\infty} n \left(a e^{-j\omega}\right)^n \qquad \text{Note } \sum_{k=0}^{\infty} k r^k = \frac{r}{(1-r)^2}$$

$$= \frac{a e^{-j\omega}}{\left(1 - a e^{-j\omega}\right)^2}$$

i: $x[n] = a^{n-1} u[n-1] \qquad |a| < 1$

$$X(\omega) = \sum_{n=1}^{\infty} a^{n-1} e^{-j\omega n} = \sum_{\ell=0}^{\infty} a^{\ell} e^{-j\omega(\ell+1)} = e^{-j\omega} \sum_{\ell=0}^{\infty} \left(a e^{-j\omega}\right)^{\ell}$$

$\ell = n-1 \qquad n = \ell + 1$

$$= \frac{e^{-j\omega}}{1 - a e^{-j\omega}}$$

5) $y[n] = 2r\cos(\theta)\, y[n-1] - r^2 y[n-2] + x[n]$ $\qquad |r| < 1$

a: Let $x_1[n] \rightarrow y_1[n]$, $x_2[n] \rightarrow y_2[n]$, $x_3[n] = a\,x_1[n] + b\,x_2[n]$

$x_3[n] \longrightarrow y_3[n]$

$$y_3[n] = 2r\cos(\theta)\, y_3[n-1] - r^2 y_3[n-2] + x_3[n]$$

$$a y_1[n] + b y_2[n] - 2r\cos(\theta)\left(a y_1[n-1] + b y_2[n-1]\right) + r^2\left(a y_1[n-2] + b y_2[n-2]\right)$$
$$= a x_1[n] + b\, x_2[n]$$

$\Rightarrow$ The system is linear

b: shifted input $x[n-n_0] \Longrightarrow y[n-n_0] = 2r\cos(\theta)\, y[n-n_0-1] - r^2 y[n-n_0-2] + x[n-n_0]$

shifted output $\Rightarrow y[n-n_0] = 2r\cos(\theta)\, y[n-n_0-1] - r^2 y[n-n_0-2] + x[n-n_0]$

$y[n-n_0] = T[x[n-n_0]] \Rightarrow$ time-invariant

c: $Y(\omega) = 2r\cos(\theta)\, e^{-j\omega} Y(\omega) - r^2 e^{-j2\omega} Y(\omega) + X(\omega)$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1 - 2r\cos(\theta) e^{-j\omega} + r^2 e^{-j2\omega}}$$

$$s = \frac{2r\cos(\theta) \pm \sqrt{4r^2\cos^2(\theta) - 4(1)(r^2)}}{2r^2} = \frac{\cos(\theta) \pm \sqrt{\cos^2\theta - 1}}{r}$$

$$= \frac{\cos(\theta) \pm \sqrt{\cos^2\theta - (\cos^2\theta + \sin^2\theta)}}{r}$$

$$= \frac{\cos\theta \pm j\sin\theta}{r}$$

$$H(\omega) = \frac{1}{\left(e^{-j\omega} - \frac{\cos\theta + j\sin\theta}{r}\right)\left(e^{-j\omega} - \frac{\cos\theta - j\sin\theta}{r}\right)}$$

$$= \frac{1}{(1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{-j\omega})}$$

d: $$H(\omega) = \frac{1}{(1 - re^{j\theta}e^{-j\omega})(1 - re^{-j\theta}e^{-j\omega})} = \frac{A}{1 - re^{j\theta}e^{-j\omega}} + \frac{B}{1 - re^{-j\theta}e^{-j\omega}}$$

$$A(1 - re^{-j\theta}e^{-j\omega}) + B(1 - re^{j\theta}e^{-j\omega}) = 1$$

$$(A + B) = 1$$

$$-re^{-j\theta}A - re^{j\theta}B = 0 \quad \longrightarrow \quad B = \frac{-re^{-j\theta}}{re^{j\theta}}A = -e^{-j2\theta}A$$

$$A + B = 1 \Rightarrow A(1 - e^{-j2\theta}) = 1 \Rightarrow A = \frac{1}{1 - e^{-j2\theta}}, \quad B = \frac{-e^{-j2\theta}}{1 - e^{-j2\theta}}$$

$$H(\omega) = \frac{1}{1 - e^{-j2\theta}} \frac{1}{1 - re^{j\theta}e^{-j\omega}} - \frac{e^{-j2\theta}}{1 - e^{-j2\theta}} \frac{1}{1 - re^{-j\theta}e^{-j\omega}}$$

$$h[n] = \frac{1}{1 - e^{-j2\theta}}(re^{j\theta})^n u[n] - \frac{e^{-j2\theta}}{1 - e^{-j2\theta}}(re^{-j\theta})^n u[n]$$

$$= \frac{r^n}{1 - e^{-j2\theta}} u[n] \left(e^{j\theta n} - e^{-j2\theta}e^{-j\theta n}\right)$$

$$= r^n u[n] \frac{e^{j\theta}}{e^{j\theta} - e^{-j\theta}} \left[e^{-j\theta}\left(e^{j\theta}e^{j\theta n} - e^{-j\theta}e^{-j\theta n}\right)\right]$$

$$= r^n u[n] \frac{1}{2j\sin(\theta)}\left(2j \sin(\theta(n+1))\right)$$

$$= r^n \frac{\sin(\theta(n+1))}{\sin(\theta)} u[n]$$