ECE 301 HW 8

1) a: $x(t) = e^{-t}u(t) \qquad y(t) = e^{-t}u(t)$

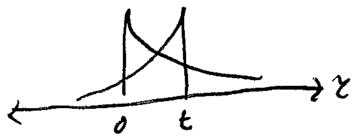
$$X(\omega) = Y(\omega) = \int_{-\infty}^{\infty} e^{-t}u(t)e^{-j\omega t}\,dt = \int_0^{\infty} e^{-t(1+j\omega)}\,dt$$

$$= \frac{-1}{1+j\omega} e^{-t(1+j\omega)}\Big|_0^{\infty} = \frac{1}{1+j\omega}$$

$$Z(\omega) = X(\omega)Y(\omega) = \frac{1}{(1+j\omega)^2}$$

$$z(t) = (x(t) * y(t))$$

for $t > 0$



$$z(t) = \int_0^t e^{-\tau}e^{-(t-\tau)}\,d\tau$$

$$= e^{-t}\int_0^t (1)\,d\tau = te^{-t}$$

So, $z(t) = te^{-t}u(t)$

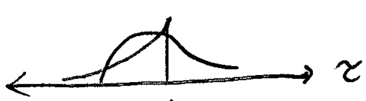
b: $x(t) = e^{-t}u(t) * e^{-t}u(t) \qquad y(t) = e^{-t}u(t)$

$$X(\omega) = \frac{1}{(1+j\omega)^2} \qquad Y(\omega) = \frac{1}{1+j\omega}$$

$$Z(\omega) = \frac{1}{(1+j\omega)^3}$$

$$z(t) = \cancel{t}\, e^{-t}u(t) * e^{-t}u(t) * e^{-t}u(t) = te^{-t}u(t) * e^{-t}u(t)$$

for $t > 0$



$$z(t) = \int_0^t \tau e^{-\tau}e^{-(t-\tau)}\,d\tau$$

$$= e^{-t}\int_0^t \tau\,d\tau = \frac{1}{2}t^2e^{-t}$$

So, $z(t) = \frac{1}{2}t^2e^{-t}u(t)$

c: $x(t) = \frac{t^{n-1}}{(n-1)!} e^{-t} u(t)$ $\qquad$ $y(t) = e^{-t} u(t)$

$$X(\omega) = \int_{-\infty}^{\infty} \frac{t^{n-1}}{(n-1)!} e^{-t} e^{-j\omega t} u(t)\,dt$$

$$= \int_0^{\infty} \frac{t^{n-1}}{(n-1)!} e^{-t(1+j\omega)}\,dt$$

$$u = \frac{t^{n-1}}{(n-1)!} \qquad dv = e^{-t(1+j\omega)}\,dt$$

$$du = \frac{t^{n-2}}{(n-2)!}\,dt \qquad v = \frac{-1}{1+j\omega} e^{-t(1+j\omega)}$$

$$= \frac{t^{n-1}}{(n-1)!}\left(\frac{-1}{1+j\omega}\right) e^{-t(1+j\omega)}\Big|_0^{\infty} + \frac{1}{1+j\omega}\int_0^{\infty} \frac{t^{n-2}}{(n-2)!} e^{-t(1+j\omega)}\,dt$$

$$= 0 - 0 + \frac{1}{1+j\omega}\int_0^{\infty} \frac{t^{n-2}}{(n-2)!} e^{-t(1+j\omega)}\,dt$$

$$= \frac{1}{(1+j\omega)^n}$$

$$Y(\omega) = \frac{1}{1+j\omega}$$

$$Z(\omega) = \frac{1}{(1+j\omega)^{n+1}}$$

$$z(t) = \frac{t^n}{n!} e^{-t} u(t)$$

d: $x(t) = e^{-t}u(t) \qquad y(t) = e^{-2t}u(t)$

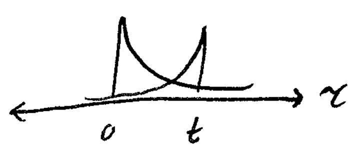
$$X(\omega) = \frac{1}{1+j\omega}$$

$$Y(\omega) = \int_0^\infty e^{-t(2+j\omega)}dt = \frac{-1}{2+j\omega} e^{-t(2+j\omega)}\Big|_0^\infty = \frac{1}{2+j\omega}$$

$$Z(\omega) = \frac{1}{1+j\omega}\,\frac{1}{2+j\omega} = \frac{1}{2+3j\omega-\omega^2}$$

$z(t) = x(t) * y(t)$

for $t>0$:



$$z(t) = \int_0^t e^{-2\tau}e^{-(t-\tau)}d\tau$$

$$= e^{-t}\int_0^t e^{-\tau}d\tau$$

$$= e^{-t}(-e^{-\tau})\Big|_0^t = e^{-t}(1-e^{-t})$$

so, $z(t) = e^{-t}(1-e^{-t})u(t)$

e: $x(t) = e^{-t}\mathcal{U}(t)$ $\qquad$ $y(t) = t e^{-2t}\mathcal{U}(t)$

$$X(\omega) = \frac{1}{1+j\omega}$$

$$Y(\omega) = \int_0^\infty t\, e^{-t(2+j\omega)}\, dt$$

$$u = t \qquad dv = e^{-t(2+j\omega)}\,\cancel{dt}\,dt$$

$$du = dt \qquad v = \frac{-1}{2+j\omega} e^{-t(2+j\omega)}$$

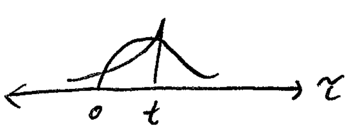
$$= \frac{-t}{2+j\omega} e^{-t(2+j\omega)}\Big|_0^\infty + \frac{1}{2+j\omega}\int_0^\infty e^{-t(2+j\omega)}\, dt$$

$$= 0 - 0 \quad \frac{-1}{(2+j\omega)^2} e^{-t(2+j\omega)}\Big|_0^\infty = \frac{1}{(2+j\omega)^2}$$

$$Z(\omega) = \frac{1}{1+j\omega}\,\frac{1}{(2+j\omega)^2}$$

$z(t) = x(t) * y(t)$

for $t > 0$:



$$z(t) = \int_0^t \tau e^{-2\tau} e^{-(t-\tau)}\, d\tau$$

$$= e^{-t}\int_0^t \tau e^{-\tau}\, d\tau$$

$$u = \tau \qquad dv = e^{-\tau} d\tau$$

$$du = d\tau \qquad v = -e^{-\tau}$$

$$= \left[-\tau e^{-\tau}\Big|_0^t + \int_0^t e^{-\tau} d\tau\right] e^{-t}$$

$$= \left[-t e^{-t} - e^{-\tau}\Big|_0^t\right] e^{-t}$$

$$= \left[-t e^{-t} - e^{-t} + 1\right] e^{-t}$$

So, $z(t) = \left(1 - e^{-t}(t+1)\right) e^{-t}\,\mathcal{U}(t)$

2) $x(t) \rightarrow \boxed{H(\omega)} \rightarrow y(t)$

a: $H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt \Rightarrow H(0) = \int_{-\infty}^{\infty} h(t)\, dt$

b: $h(t) = \int_{-\infty}^{\infty} H(\omega)\, e^{j\omega t}\, d\omega \Rightarrow h(0) = \int_{-\infty}^{\infty} H(\omega)\, d\omega$

c: $x(t) = \alpha = \alpha e^{j(0)t}$

$y(t) = \alpha H(0)$

d: $x(t) = \alpha \Rightarrow y(t) = \alpha H(0)$

$H(0) = \int_{-\infty}^{\infty} h(t)\, dt \Rightarrow y(t) = \alpha \int_{-\infty}^{\infty} h(t)\, dt$

e: DC gain of $A \Rightarrow H(0) = \int_{-\infty}^{\infty} h(t)\, dt = A$

$\int_{-\infty}^{\infty} h(t)\, dt = A$

f: $H(0) = A$

3) $T\left[e^{-2t}u(t)\right] = te^{-t}u(t) + 2e^{-2t}u(t)$

$x(t) = e^{-2t}u(t) \Leftrightarrow X(\omega) = \frac{1}{2+j\omega}$

$y(t) = te^{-t}u(t) + 2e^{-2t}u(t)$

$e^{-2t}u(t) \Leftrightarrow \frac{1}{2+j\omega}$

$te^{-t}u(t) \Leftrightarrow j\frac{d}{d\omega}\frac{1}{1+j\omega} = j\frac{-j}{(1+j\omega)^2} = \frac{1}{(1+j\omega)^2}$

$y(t) \Leftrightarrow Y(\omega) = \frac{1}{(1+j\omega)^2} + \frac{2}{2+j\omega} = \frac{2+j\omega + 2(1+j\omega)^2}{(2+j\omega)(1+j\omega)^2}$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\frac{2+j\omega+2(1+j\omega)^2}{(2+j\omega)(1+j\omega)^2}}{\frac{1}{2+j\omega}} = \frac{2+j\omega+2(1+j\omega)^2}{(1+j\omega)^2} = 2 + \frac{2+j\omega}{(1+j\omega)^2}$$

4) $\frac{d^2}{dt^2} y(t) + 3\frac{d}{dt} y(t) + y(t) = \frac{d}{dt} x(t) + x(t)$

a: $(j\omega)^2 Y(\omega) + 3j\omega Y(\omega) + Y(\omega) = j\omega X(\omega) + X(\omega)$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1+j\omega}{1+3j\omega+(j\omega)^2}$$

b: $s^2+3s+1 = 0 \Rightarrow s = \frac{-3 \pm \sqrt{9-4}}{2(1)} = -3 \pm \sqrt{5}$

$$H(\omega) = \frac{1+j\omega}{(-3+\sqrt{5}-j\omega)(-3-\sqrt{5}-j\omega)}$$

5) $\frac{d^2}{dt^2} y(t) + 7\frac{d}{dt} y(t) + 10 y(t) = \frac{d}{dt} x(t) - x(t)$

a: $(j\omega)^2 Y(\omega) + 7j\omega Y(\omega) + 10 Y(\omega) = j\omega X(\omega) - X(\omega)$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{-1+j\omega}{10 + 7j\omega - \omega^2}$$

$$= \frac{-1+j\omega}{(j\omega+5)(j\omega+2)}$$

b: $$\frac{A}{j\omega+5} + \frac{B}{2+j\omega} = \frac{-1+j\omega}{(j\omega+5)(j\omega+2)}$$

$A(j\omega+2) + B(j\omega+5) = -1 + j\omega$

$j\omega(A+B) = 1$ $\qquad B = 1 - A$

$2A + 5B = -1$ $\qquad 2A + 5(1-A) = -1$

$-3A = -6 \Rightarrow A = 2 \qquad B = -1$

$$H(\omega) = \frac{2}{5+j\omega} + \frac{(-1)}{2+j\omega}$$

$h(t) = 2e^{-5t}u(t) - e^{-2t}u(t)$

c: $x(t) = e^{-t}u(t) \Leftrightarrow X(\omega) = \frac{1}{1+j\omega}$

$$Y(\omega) = H(\omega)X(\omega) = \frac{-1+j\omega}{(5+j\omega)(2+j\omega)(1+j\omega)}$$

$$\frac{A}{5+j\omega} + \frac{B}{2+j\omega} + \frac{C}{1+j\omega} = Y(\omega)$$

$A(2+j\omega)(1+j\omega) + B(5+j\omega)(1+j\omega) + C(5+j\omega)(2+j\omega) = -1+j\omega$

$A(2+3j\omega+(j\omega)^2) + B(5+6j\omega+(j\omega)^2) + C(10+7j\omega+(j\omega)^2) = -1+j\omega$

$(j\omega)^2(A+B+C) = 0$

$j\omega(3A+6B+7C) = 1$

$2A+5B+10C = -1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 6 & 7 \\ 2 & 5 & 10 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix}$$

$$Y(\omega) = -\frac{1}{2} \frac{1}{5+j\omega} + \frac{1}{2+j\omega} - \frac{1}{2} \frac{1}{1+j\omega}$$

$$y(t) = -\frac{1}{2} e^{-5t} u(t) + e^{-2t} u(t) - \frac{1}{2} e^{-t} u(t)$$

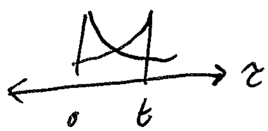
5) a: $X(\omega) = \dfrac{1}{5+j\omega}$

$x(t) = e^{-5t}u(t)$

b: $X(\omega) = \dfrac{1}{(5+j\omega)^2}$

$x(t) = e^{-5t}u(t) * e^{-5t}u(t)$

for $t>0$:



$$x(t) = \int_0^t e^{-5\tau} e^{-5(t-\tau)}\, d\tau$$

$$= e^{-5t}\int_0^t (1)\, d\tau = t e^{-5t}$$

$x(t) = t e^{-5t}u(t)$

c: $X(\omega) = \dfrac{1}{(5+j\omega)(2+j\omega)} = \dfrac{A}{5+j\omega} + \dfrac{B}{2+j\omega}$

$A(2+j\omega) + B(5+j\omega) = 1$

$j\omega(A+B) = 0$ $\qquad B = -A$

$2A + 5B = 1$ $\qquad -3A = 1 \Rightarrow A = \frac{-1}{3} \qquad B = \frac{1}{3}$

$x(t) = -\frac{1}{3}e^{-5t}u(t) + \frac{1}{3}e^{-2t}u(t)$