ECE 301 HW 7

1) $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

a: $x(t) = e^{-at} u(t) \quad a > 0$

$$X(\omega) = \int_0^\infty e^{-at} e^{-j\omega t} dt = \int_0^\infty e^{-t(a+j\omega)} dt$$

$$= \frac{-1}{a+j\omega} e^{-t(a+j\omega)} \Big|_0^\infty = \frac{1}{a+j\omega}$$

b: $x(t) = t e^{-at} u(t) \quad a > 0$

$$X(\omega) = \int_0^\infty t e^{-at} e^{-j\omega t} dt = \int_0^\infty t e^{-t(a+j\omega)} dt$$

$u = t \qquad dv = e^{-t(a+j\omega)} dt$

$du = dt \qquad v = \frac{-1}{a+j\omega} e^{-t(a+j\omega)}$

$$= \frac{-t}{a+j\omega} e^{-t(a+j\omega)} \Big|_0^\infty + \frac{1}{a+j\omega} \int_0^\infty e^{-t(a+j\omega)} dt$$

$$= 0 - 0 - \frac{1}{(a+j\omega)^2} e^{-t(a+j\omega)} \Big|_0^\infty = \frac{1}{(a+j\omega)^2}$$

c: $x(t) = rect(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & ow \end{cases}$

$$X(\omega) = \int_{-\frac{1}{2}}^{\frac{1}{2}} (1) e^{-j\omega t} dt = \frac{-1}{j\omega} e^{-j\omega t} \Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{-1}{j\omega} \left(e^{-j\frac{1}{2}\omega} - e^{j\frac{1}{2}\omega} \right)$$

$$= \frac{2}{\omega} \sin\left(\tfrac{1}{2}\omega\right) = \frac{\sin\left(\frac{1}{2}\omega\right)}{\frac{1}{2}\omega} = \operatorname{sinc}\left(\tfrac{1}{2\pi}\omega\right)$$

d: $x(t) = rect\left(\frac{t-a}{b}\right) = \begin{cases} 1 & |t-a| < \frac{b}{2} \\ 0 & ow \end{cases}$

$$X(\omega) = \int_{a-\frac{b}{2}}^{a+\frac{b}{2}} (1) e^{-j\omega t} dt = \frac{-1}{j\omega} e^{-j\omega t} \Big|_{a-\frac{b}{2}}^{a+\frac{b}{2}} = \frac{-1}{j\omega} \left(e^{-j\omega(a+\frac{b}{2})} - e^{-j\omega(a-\frac{b}{2})} \right)$$

$$= \frac{2}{\omega} e^{-j\omega a} \sin\left(\tfrac{b}{2}\omega\right)$$

e: $x(t) = \delta(t)$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{-j\omega t}\Big|_{t=0} = 1$$

f: $x(t) = a\delta(t-b)$

$$X(\omega) = \int_{-\infty}^{\infty} a\,\delta(t-b)\, e^{-j\omega t} dt = a\, e^{-j\omega t}\Big|_{t=b} = a\, e^{-j\omega b}$$

2) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$

a: $X(\omega) = \delta(\omega)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\, e^{j\omega t}\Big|_{\omega=0} = \frac{1}{2\pi}$$

b: $X(\omega) = \delta(\omega - \omega_0)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\, e^{j\omega t}\Big|_{\omega=\omega_0} = \frac{1}{2\pi}\, e^{j\omega_0 t}$$

c: $X(\omega) = \text{rect}(\omega) = \begin{cases} 1 & |\omega| < \frac{1}{2} \\ 0 & \text{ow} \end{cases}$

$$x(t) = \frac{1}{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} (1)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\,\frac{1}{jt}\, e^{j\omega t}\Big|_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \frac{1}{2\pi}\,\frac{1}{jt}\left(e^{j\frac{1}{2}t} - e^{-j\frac{1}{2}t}\right) = \frac{1}{\pi t}\sin\left(\tfrac{1}{2}t\right)$$

3) a: $x(t) = \mathrm{sinc}(t)$

$$= \frac{\sin(\pi t)}{\pi t} \Longleftrightarrow X(\omega) = \mathrm{rect}\left(\frac{\omega}{2\pi}\right)$$

$$= \begin{cases} 1 & |\omega| < \pi \\ 0 & |\omega| > \pi \end{cases}$$

b: $x(t) = \mathrm{sinc}\left(\frac{t-a}{b}\right) = \mathrm{sinc}\left(\frac{t}{b} - \frac{a}{b}\right)$

Let $x_1(t) = \mathrm{sinc}\left(\frac{t}{b}\right)$ $\quad X_1(\omega) = |b|\,\mathrm{rect}\left(\frac{b}{2\pi}\omega\right)$

$x(t) = x_1\left(t - \frac{a}{b}\right)$ $\quad X(\omega) = e^{-j\omega\frac{a}{b}} X_1(\omega)$

$$= e^{-j\omega\frac{a}{b}}\,|b|\,\mathrm{rect}\left(\frac{b}{2\pi}\omega\right)$$

c: $x(t) = 1 \Longleftrightarrow X(\omega) = 2\pi\delta(\omega)$

d: $x(t) = e^{j\omega_0 t} \Longleftrightarrow X(\omega) = 2\pi\delta(\omega - \omega_0)$

e: $x(t) = \cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$

$$\Longleftrightarrow X(\omega) = \cancel{\tfrac{1}{2}}\pi\,\delta(\omega - \omega_0) + \cancel{\tfrac{1}{2}}\pi\,\delta(\omega + \omega_0)$$

f: $x(t) = \sin(\omega_0 t) = \frac{1}{2j}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$

$$\Longleftrightarrow \cancel{X(\omega) = \tfrac{1}{2j}} \quad X(\omega) = \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$

4) $x(t) \Leftrightarrow X(\omega)$ $\quad y(t) \Leftrightarrow Y(\omega)$

a: $5x(t-a) \Leftrightarrow 5e^{-j\omega a} X(\omega)$

b: $X(t) \Leftrightarrow 2\pi x(-\omega)$

c: $x(t) * y(t) \Leftrightarrow X(\omega) Y(\omega)$

d: $x(t) y(t) \Leftrightarrow \frac{1}{2\pi} X(\omega) * Y(\omega)$

e: ~~xhc~~ $x(-t) \Leftrightarrow X(-\omega)$

f: $x(t) e^{j\omega_0 t} \Leftrightarrow \frac{1}{2\pi} X(\omega) * 2\pi\delta(\omega-\omega_0) = X(\omega-\omega_0)$

g: $\frac{1}{|a|} X(\frac{\omega}{a}) \Leftrightarrow x(at)$

5) $x(t) \Leftrightarrow X(\omega) \quad y(t) \Leftrightarrow Y(\omega)$

b: $\int_{-\infty}^{\infty} x(-t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(u) e^{j\omega u} du = \int_{-\infty}^{\infty} x(t) e^{-j(-\omega)t} dt = X(-\omega)$

$u = -t \quad du = -dt$

Thus $x(-t) \Leftrightarrow X(-\omega)$

c: $\int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(u) e^{-j\omega(u+t_0)} du = e^{-j\omega t_0} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$u = t - t_0 \quad du = dt \qquad = e^{-j\omega t_0} X(\omega)$

thus $x(t-t_0) \Leftrightarrow e^{-j\omega t_0} X(\omega)$

d: $\int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt = \begin{cases} \int_{-\infty}^{\infty} x(u) e^{-j\omega \frac{u}{a}} \frac{1}{a} du & a > 0 \\ \int_{\infty}^{-\infty} x(u) e^{-j\omega \frac{u}{a}} \frac{1}{a} du & a < 0 \end{cases}$

$u = at \quad du = a\,dt$

$= \frac{1}{|a|} X\left(\frac{\omega}{a}\right) \qquad$ Thus $x(at) \Leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

e: $X^*(-\omega) = \left(\int_{-\infty}^{\infty} x(t) e^{j\omega t} dt \right)^* = \int_{-\infty}^{\infty} x^*(t) e^{-j\omega t} dt$

$= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ if $x(t)$ is real $(x^*(t) = x(t))$

$= X(\omega)$

h: $\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2\pi} X(\omega) * Y(\omega) e^{j\omega t} d\omega = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(\omega - \tau) Y(\tau) d\tau \, e^{j\omega t} d\omega$

$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(\omega - \tau) Y(\tau) e^{j\omega t} d\omega \, d\tau \qquad u = \omega - \tau \quad du = d\omega$

$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(u) Y(\tau) e^{jt(u+\tau)} du \, d\tau$

$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(u) e^{jut} Y(\tau) e^{j\tau t} du \, d\tau$

$= \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \right) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{j\omega t} d\omega \right) = x(t) y(t)$

$\cancel{j(: \frac{d}{dt}\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}$

j : $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{-j\omega t}\, d\omega$

$$\frac{d}{dt}\, x(t) = \frac{1}{2\pi}\, \frac{d}{dt}\int_{-\infty}^{\infty} X(\omega)\, e^{+j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, \frac{d}{dt} e^{+j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left(j\omega\, e^{j\omega t}\right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega\, X(\omega)\, e^{j\omega t}\, d\omega$$

Thus $\frac{dx(t)}{dt} \Longleftrightarrow j\omega\, X(\omega)$

6) $X^*(-\omega) = X(\omega) \iff x(t)$ is purely real

$X^*(-\omega) = -X(\omega) \iff x(t)$ is purely imaginary

$X(-\omega) = X(\omega) \iff x(t)$ is even

$X(-\omega) = -X(\omega) \iff x(t)$ is odd

a: $X(\omega) = \sin(2\omega)\cos(3\omega)$

$X^*(-\omega) = \sin(-2\omega)\cos(-3\omega) = -\sin(2\omega)\cos(3\omega) = -X(\omega)$

Thus, $x(t)$ is purely imaginary

$X(-\omega) = -X(\omega)$

Thus, $x(t)$ is odd

b: $X(\omega) = \sin(\omega)\, e^{j(2\omega + \frac{\pi}{2})}$

$X^*(-\omega) = \sin(-\omega)\, e^{-j(-2\omega + \frac{\pi}{2})}$

$= -\sin(\omega)\, e^{j(2\omega - \frac{\pi}{2})}$

$X^*(-\omega) \neq \pm X(\omega) \iff x(t)$ is complex

$X(-\omega) = \sin(-\omega)\, e^{j(-2\omega + \frac{\pi}{2})} = -\sin(\omega)\, e^{j(-2\omega + \frac{\pi}{2})}$

$X(-\omega) \neq \pm X(\omega) \iff x(t)$ is neither even nor odd

c: $X(\omega) = \mathcal{U}(\omega) - \mathcal{U}(\omega - 4\pi)$

$X^*(-\omega) = \mathcal{U}(-\omega) - \mathcal{U}(-\omega - 4\pi) = \mathcal{U}(\omega + 4\pi) - \mathcal{U}(\omega)$

$X^*(-\omega) \neq \pm X(\omega) \iff x(t)$ is complex

$X(-\omega) = \mathcal{U}(\omega + 4\pi) - \mathcal{U}(\omega)$

$\neq \pm X(\omega) \iff x(t)$ is neither even nor odd

7) a: $x(t) = \frac{1}{5+j2\pi t} = \frac{\frac{1}{2\pi}}{\frac{5}{2\pi}+jt}$

$$f(t) = e^{-at}u(t) \Leftrightarrow g(\omega) = \frac{1}{a+j\omega} \qquad g(t) \Leftrightarrow 2\pi f(-\omega)$$

$$x(t) = \frac{1}{2\pi} g(t) \qquad a = \frac{5}{2\pi}$$

$$X(\omega) = \frac{1}{2\pi} 2\pi f(-\omega) = e^{-\frac{5}{2\pi}(-\omega)}u(-\omega) = e^{\frac{5}{2\pi}\omega}u(-\omega)$$

b: We know $e^{-|t|} \Leftrightarrow \frac{2}{1+\omega^2}$ and $t\,x(t) \Leftrightarrow j\frac{d}{d\omega}X(\omega)$

So, $te^{-|t|} \Leftrightarrow j\frac{d}{d\omega}\frac{2}{1+\omega^2} = \frac{-4j\omega}{(1+\omega^2)^2}$

Let $f(t) = te^{-|t|}$ $\qquad g(\omega) = \frac{-4j\omega}{(1+\omega^2)^2}$

$$x(t) = \frac{t}{(1+t^2)^2} = \frac{-1}{4j} g(t)$$

$$X(\omega) = \frac{-1}{4j} 2\pi f(-\omega) = \frac{j\pi}{2}(-\omega)e^{-|-\omega|} = \frac{-j\pi\omega}{2} e^{-|\omega|}$$