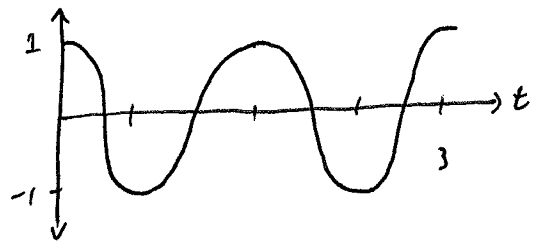
ECE 301 HW6 Solutions

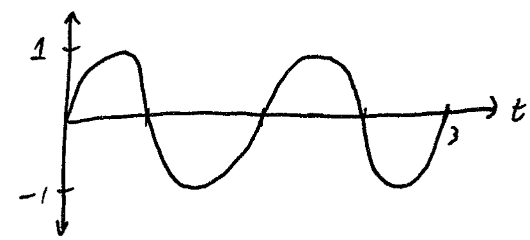
1) a: $x(t) = e^{j\frac{2\pi}{3}t}$ $\qquad T=3$

$= \cos\left(\frac{2\pi}{3}t\right) + j\sin\left(\frac{2\pi}{3}t\right)$

$\Re\{x(t)\}$



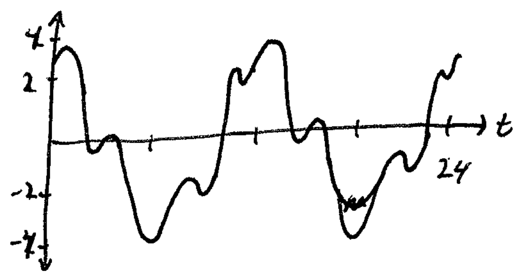
$\Im\{x(t)\}$



$$a_k = \frac{1}{T}\int_0^3 e^{j\frac{2\pi}{3}t} e^{-j\frac{2\pi}{3}kt}\,dt = \begin{cases} \cancel{3}1 & k=1 \\ 0 & k\neq 1 \end{cases} \qquad a_k = \cancel{3}\delta[k-1]$$

b: $x(t) = \sin\left(\frac{2\pi}{3}t\right) + 3\cos\left(\frac{\pi}{6}t\right)$ $\qquad T=12$

$\Re\{x(t)\} = x(t)$ $\qquad \Im\{x(t)\} = 0$

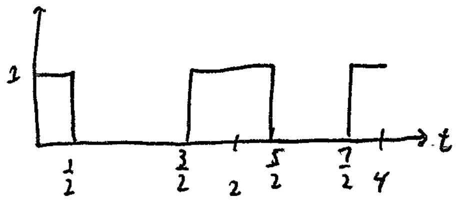


$$x(t) = \frac{1}{2j}\left(e^{j\frac{2\pi}{3}t} - e^{-j\frac{2\pi}{3}t}\right) + \frac{3}{2}\left(e^{j\frac{\pi}{6}t} + e^{-j\frac{\pi}{6}t}\right)$$

$$= \sum_k a_k e^{j\frac{\pi}{6}kt} \qquad \Rightarrow \qquad a_k = \begin{cases} \frac{3}{2} & k=\pm 1 \\ \frac{1}{2j} & k=4 \\ -\frac{1}{2j} & k=-4 \\ 0 & \text{ow} \end{cases}$$

c: $x(t) = \text{rect}(t) \qquad T = 2$

$\Re\{x(t)\} = x(t) \qquad \Im\{x(t)\} = 0$



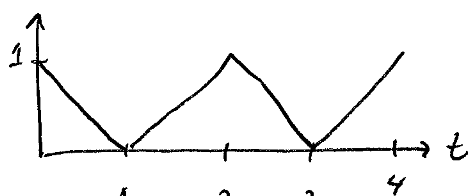
$$a_k = \frac{1}{2}\int_{-1}^{1} x(t)\, e^{-j\pi k t}\, dt$$

$$= \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} (1)\, e^{-j\pi k t} = \frac{1}{2}\left(\frac{-1}{j\pi k} e^{-j\pi k t}\right)\Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{-1}{j2\pi k}\left(e^{-j\frac{\pi}{2}k} - e^{j\frac{\pi}{2}k}\right)$$

$$= \frac{1}{\pi k}\,\frac{1}{j2}\left(e^{j\frac{\pi}{2}k} - e^{-j\frac{\pi}{2}k}\right) = \frac{\sin\left(\frac{\pi}{2}k\right)}{\pi k}$$

d: $x(t) = \Lambda(t) \qquad T = 2$

$\Re\{x(t)\} = x(t) \qquad \Im\{x(t)\} = 0$



$$a_k = \frac{1}{2}\int_{-1}^{1} \Lambda(t)\, e^{-j\pi k t}\, dt = \frac{1}{2}\int_{-1}^{0} (t+1)\, e^{-j\pi k t}\, dt + \frac{1}{2}\int_{0}^{1} (1-t)\, e^{-j\pi t k}\, dt$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-j\pi k t} + \frac{1}{2}\int_{-1}^{0} (t)\, e^{-j\pi k t}\, dt + \frac{1}{2}\int_{0}^{1} (-t)\, e^{-j\pi k t}\, dt$$

$$u = t \qquad dv = e^{-j\pi k t}\, dt$$
$$du = dt \qquad v = \frac{-1}{j\pi k} e^{-j\pi k t}$$

$$= \frac{1}{2}\,\frac{-1}{j\pi k} e^{-j\pi k t}\Big|_{-1}^{1} + \frac{1}{2}\left[\frac{-t}{j\pi k} e^{-j\pi k t}\Big|_{-1}^{0} + \frac{1}{j\pi k}\int_{-1}^{0} e^{-j\pi k t}\, dt\right]$$

$$- \frac{1}{2}\left[\frac{-t}{j\pi k} e^{-j\pi k t}\Big|_{0}^{1} + \frac{1}{j\pi k}\int_{0}^{1} e^{-j\pi k t}\right]$$

$$= \frac{1}{2}\,\frac{-1}{j\pi k}\left(e^{-j\pi k} - e^{j\pi k}\right) + \frac{1}{2}\left[\frac{-1}{j\pi k} e^{j\pi k} - \frac{1}{(j\pi k)^2} e^{-j\pi k t}\Big|_{-1}^{0}\right]$$

$$- \frac{1}{2}\left[\frac{-1}{j\pi k} e^{-j\pi k} - \frac{1}{(j\pi k)^2} e^{-j\pi k t}\Big|_{0}^{1}\right]$$

$$a_k = \frac{1}{\pi k} \sin(\pi k) + \frac{1}{2}\left[\frac{-1}{j\pi k} e^{+j\pi k} + \frac{1}{j\pi k} e^{-j\pi k} + \frac{1}{(\pi k)^2}\left(1 - e^{j\pi k} - e^{-j\pi k} + 1 \right)\right]$$

$$= \delta[k] + \frac{1}{2}\left[\frac{-1}{j\pi k}\left(2j \sin(\pi k)\right) + \frac{1}{(\pi k)^2}\left(2 - 2\cos(\pi k)\right)\right]$$

$$= \delta[k] - \frac{1}{\pi k} \sin(\pi k) + \frac{1}{(\pi k)^2}\left(1 - \cos(\pi k)\right)$$

$$= \frac{1}{(\pi k)^2}\left(1 - (-1)^k\right)$$

2) $x(t) = \sum_k a_k e^{j\frac{2\pi}{T}kt}$

a: $y(t) = \frac{d}{dt} x(t) = \frac{d}{dt} \sum_k a_k e^{j\frac{2\pi}{T}kt}$

$= \sum_k a_k \frac{d}{dt}\left(e^{j\frac{2\pi}{T}kt}\right) = \sum_k a_k \left(j\frac{2\pi}{T}k\right) e^{j\frac{2\pi}{T}kt}$

$y(t) = \sum_k b_k e^{j\frac{2\pi}{T}kt} \Rightarrow b_k = jk\frac{2\pi}{T} a_k$

b: $y(t) = x(-t) = \sum_k a_k e^{j\frac{2\pi}{T}k(-t)}$

$= \sum_k a_{-k} e^{j\frac{2\pi}{T}kt} \Rightarrow b_k = a_{-k}$

c: $x(t)$ real $\Leftrightarrow x^*(t) = x(t)$

$x^*(t) = \left(\sum_k a_k e^{j\frac{2\pi}{T}kt}\right)^* = \sum_k a_k^* e^{-j\frac{2\pi}{T}kt} = \sum_k a_{-k}^* e^{j\frac{2\pi}{T}kt}$

$x^*(t) = x(t) \Rightarrow \sum_k a_{-k}^* e^{j\frac{2\pi}{T}kt} = \sum_k a_k e^{j\frac{2\pi}{T}kt} \Rightarrow a_{-k}^* = a_k$

d: $x(t)$ is real and $x(t) = x(-t)$

From parts b & c, $a_k = a_{-k}$ & $a_{-k}^* = a_k$

$a_k = a_{-k} = a_k^* \Rightarrow a_k$ are real

3) $x(t) = \sum_k a_k e^{j\frac{2\pi}{T}kt}$

a: $T=2$, $a_k = \left(\frac{1}{2}\right)^{|k|}$

$$x(t) = \sum_k \left(\tfrac{1}{2}\right)^{|k|} e^{j\pi kt} = \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k e^{j\pi kt} + \sum_{k=-\infty}^{0} \left(\tfrac{1}{2}\right)^{-k} e^{j\pi kt} - \left(\tfrac{1}{2}\right)^0 e^{j\pi t(0)}$$

$$= \sum_{k=0}^{\infty} \left(\tfrac{1}{2} e^{j\pi t}\right)^k + \sum_{k=0}^{\infty} \left(\tfrac{1}{2} e^{-j\pi t}\right)^k - 1$$

$$= \frac{1}{1-\frac{1}{2}e^{j\pi t}} + \frac{1}{1-\frac{1}{2}e^{-j\pi t}} - 1$$

$$= \frac{1-\frac{1}{2}e^{-j\pi t} + 1 - \frac{1}{2}e^{j\pi t}}{1-\frac{1}{2}e^{-j\pi t} - \frac{1}{2}e^{j\pi t} + \frac{1}{4}} - 1$$

$$= \frac{2-\cos(\pi t)}{\frac{5}{4}-\cos(\pi t)} - 1 = \frac{2-\cos(\pi t) - \frac{5}{4} + \cos(\pi t)}{\frac{5}{4}-\cos(\pi t)}$$

$$= \frac{\frac{3}{4}}{\frac{5}{4}-\cos(\pi t)} = \frac{3}{5-4\cos(\pi t)}$$

b: $T=4$ $\quad a_k = \begin{cases} jk & |k|<3 \\ 0 & \text{ow} \end{cases}$

$$x(t) = j(-2)e^{-j\frac{\pi}{2}t(2)} + j(-1)e^{-j\frac{\pi}{2}t} + j(0)e^{j\frac{\pi}{2}t(0)} + j(1)e^{+j\frac{\pi}{2}t} + j(2)e^{j\frac{\pi}{2}t(2)}$$

$$= j\left(2j\sin\left(\tfrac{\pi}{2}t\right)\right) + 2j\left(2j\sin(\pi t)\right)$$

$$= -4\sin(\pi t) - 2\sin\left(\tfrac{\pi}{2}t\right)$$

c: $a_k = \cos(\frac{\pi}{4}k)$ $\qquad T = 4$

$$= \frac{1}{2}e^{j\frac{\pi}{4}k} + \frac{1}{2}e^{-j\frac{\pi}{4}k}$$

$$e^{j\frac{\pi}{4}k} \Longleftrightarrow \delta(t-\tfrac{1}{2}) \qquad \& \qquad e^{-j\frac{\pi}{4}k} \Longleftrightarrow \delta(t+\tfrac{1}{2})$$

$$x(t) = \frac{1}{2}\delta(t-\tfrac{1}{2}) + \frac{1}{2}\delta(t+\tfrac{1}{2})$$

4) $x(t) = \sum_k a_k e^{j\frac{2\pi}{T}kt}$ $\quad y(t) = x(t) * h(t)$ $\quad x(t+\ell T) = x(t)$ $\quad \ell \in \mathbb{Z}$

a: $y(t+\ell T) = x(t+\ell T) * h(t)$ since the system is LTI

$= x(t) * h(t)$ since $x(t)$ is periodic

$= y(t)$

$y(t+\ell T) = y(t) \Rightarrow y(t)$ is periodic with period T

b: $y(t) = x(t) * h(t)$

$$= \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau = \int_{-\infty}^{\infty} h(t-\tau) \sum_k a_k e^{j\frac{2\pi}{T}k\tau}\, d\tau$$

$$= \sum_k a_k \int_{-\infty}^{\infty} h(t-\tau) e^{j\frac{2\pi}{T}k\tau}\, d\tau$$

$$= \sum_k a_k \int_{-\infty}^{\infty} h(\tau) e^{j\frac{2\pi}{T}k(t-\tau)}\, d\tau = \sum_k a_k e^{j\frac{2\pi}{T}kt} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-j\frac{2\pi}{T}k\tau}\, d\tau}_{c_k}$$

$b_k = a_k c_k$

c: $c_k = \int_{-\infty}^{\infty} h(t)\, e^{-j\frac{2\pi}{T}kt}\, dt$

~~The Fourier Series [illegible]~~