ECE 301 HW 5 Solutions

1) a: $\langle \sin(m\omega_0 t), \sin(n\omega_0 t)\rangle = \int_0^T \sin(m\omega_0 t)\sin(n\omega_0 t)\,dt \qquad m \neq n$

$$= \int_0^{\frac{2\pi}{\omega_0}} \frac{1}{2j}\left(e^{jm\omega_0 t} - e^{-jm\omega_0 t}\right)\frac{1}{2j}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)dt$$

$$= \left(\frac{1}{2j}\right)^2 \int_0^{\frac{2\pi}{\omega_0}}\left[e^{j\omega_0 t(m+n)} - e^{j\omega_0 t(m-n)} - e^{j\omega_0 t(n-m)} + e^{-j\omega_0 t(n+m)}\right]dt$$

$$= \frac{-1}{4}\int_0^{\frac{2\pi}{\omega_0}}\left[2\cos(\omega_0(n+m)t) - 2\cos(\omega_0(n-m)t)\right]dt$$

$$= -\frac{1}{2}\left[\frac{1}{\omega_0(n+m)}\sin(\omega_0(n+m)t) - \frac{1}{\omega_0(n-m)}\sin(\omega_0(n-m)t)\right]_0^{\frac{2\pi}{\omega_0}} = 0$$

Thus, they are orthogonal

b: $\langle x_e(t), x_o(t)\rangle = \int_{-T}^{T}\left(\frac{x(t)+x(-t)}{2}\right)\left(\frac{x(t)-x(t)}{2}\right)dt$

$$= \int_{-T}^{T}\frac{1}{4}\left[x^2(t) - x(t)x(-t) + x(-t)x(t) - x^2(-t)\right]dt$$

$$= \frac{1}{4}\left[\int_{-T}^{T} x^2(t)\,dt \mp \int_{-T}^{T} x^2(-t)\,dt\right]$$

Let $U = -t \quad dU = -dt$

$$= \frac{1}{4}\left[\int_{-T}^{T} x^2(t)\,dt \mp \int_{T}^{-T} x^2(U)(-dU)\right]$$

$$= \frac{1}{4}\left[\int_{-T}^{T} x^2(t)\,dt \mp \int_{-T}^{T} x^2(U)\,dU\right] = 0$$

Thus, they are orthogonal for any T

c: $f(t) = t \qquad g(t) = t - \alpha t^2$

$\langle f(t), g(t) \rangle = 0$ for $t \in (0,1) \Rightarrow \int_0^1 f(t) g(t)\,dt = 0$

$$\int_0^1 t(t - \alpha t^2)\,dt = \int_0^1 (t^2 - \alpha t^3)\,dt$$

$$= \frac{1}{3}t^3 - \frac{1}{4}\alpha t^4 \Big|_0^1 = \frac{1}{3} - \frac{1}{4}\alpha \quad \Rightarrow \quad \alpha = \frac{4}{3}$$

$\langle \frac{f(t)}{c_1}, \frac{f(t)}{c_1} \rangle = 1 \Rightarrow \frac{1}{c_1^2}\int_0^1 t^2\,dt = 1$

$$c_1^2 = \frac{1}{3}t^3 \Big|_0^1 = \frac{1}{3} \quad \Rightarrow \quad c_1 = \frac{1}{\sqrt{3}}$$

$\langle \frac{g(t)}{c_2}, \frac{g(t)}{c_2} \rangle = 1 \Rightarrow \frac{1}{c_2^2}\int_0^1 (t - \frac{4}{3}t^2)^2\,dt = 1$

$$c_2^2 = \int_0^1 (t^2 - \frac{8}{3}t^3 + \frac{16}{9}t^4)\,dt = \left[\frac{1}{3}t^3 - \frac{2}{3}t^4 + \frac{16}{45}t^5\right]_0^1$$

$$= \frac{1}{3} - \frac{2}{3} + \frac{16}{45} = \frac{16}{45} - \frac{1}{3} = \frac{16 - 15}{45} = \frac{1}{45}$$

$$\Rightarrow c_2 = \frac{1}{\sqrt{45}} = \frac{1}{3\sqrt{5}}$$

2) a: $\phi_k(t) = \frac{1}{\sqrt{T}} e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T} \qquad t \in (0, T)$

$$\langle \phi_k(t), \phi_m(t) \rangle = \frac{1}{T}\int_0^T e^{jk\omega_0 t} e^{-jm\omega_0 t}\,dt = \frac{1}{T}\int_0^T e^{j\frac{2\pi}{T}(k-m)t}\,dt$$

$$= \frac{1}{T}\frac{T}{2\pi(k-m)} e^{j\frac{2\pi}{T}(k-m)t}\Big|_0^T = \frac{1}{2\pi(k-m)}\left(e^{j2\pi(k-m)} - 1\right) = 0$$

$$\cancel{\langle \phi_k(t), \phi_k(t) \rangle = \frac{1}{T}\int_0^T e^{j2k\omega_0 t}\,dt = \frac{1}{T}\frac{1}{2k\omega_0} e^{j2k\omega_0 t}\Big|_0^T}$$

$$\cancel{= \frac{1}{T}\frac{T}{4\pi k}\left(e^{j4\pi k} - 1\right)}$$

$$\langle \phi_k(t), \phi_k(t) \rangle = \frac{1}{T}\int_0^T e^{jk\omega_0 t} e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_0^T (1)\,dt = 1$$

$\Rightarrow \{\phi_k(t)\}$ is orthonormal b: $\phi_k(t) = \cos(k\omega_0 t) \qquad \omega_0 = \frac{2\pi}{T} \qquad t \in (0, T)$

$$\langle \phi_k(t), \phi_m(t) \rangle = \int_0^T \cos(k\omega_0 t)\cos(m\omega_0 t)\,dt$$

$$= \int_0^T \tfrac{1}{2}\left(e^{jk\omega_0 t} + e^{-jk\omega_0 t}\right)\tfrac{1}{2}\left(e^{jm\omega_0 t} + e^{-jm\omega_0 t}\right)dt$$

$$= \frac{1}{4}\int_0^T \left[e^{j\omega_0 t(k+m)} + e^{j\omega_0 t(k-m)} + e^{j\omega_0 t(m-k)} + e^{-j\omega_0 t(m+k)}\right]dt$$

$$= \frac{1}{4}\int_0^T \left[2\cos(\omega_0(k+m)t) + 2\cos(\omega_0(k-m)t)\right]dt$$

$$= \frac{1}{2}\left[\frac{1}{\omega_0(k+m)}\sin(\omega_0(k+m)t) + \frac{1}{\omega_0(k-m)}\sin(\omega_0(k-m)t)\right]_0^T$$

$$= \frac{1}{2}\left[\frac{1}{\omega_0(k+m)}\left(\sin(2\pi(k+m)) - \sin(0)\right) + \frac{1}{\omega_0(k-m)}\left(\sin(2\pi(k-m)) - \sin(0)\right)\right]$$

$$= 0 \quad \Rightarrow \text{orthogonal}$$

$$\langle \phi_k(t), \phi_k(t) \rangle = \int_0^T \cos^2(k\omega_0 t)\,dt = \int_0^T \tfrac{1}{2}\left(1 + \cos(2k\omega_0 t)\right)dt$$

$$= \left[\tfrac{1}{2}t + \frac{1}{2k\omega_0}\sin(2k\omega_0 t)\right]_0^T = \tfrac{1}{2}T + \frac{1}{2k\omega_0}\left(\sin(4\pi k) - \sin(0)\right) = \tfrac{1}{2}T$$

for $\left\langle \frac{\phi_k(t)}{c}, \frac{\phi_k(t)}{c} \right\rangle = 1, \qquad c = \sqrt{\frac{T}{2}}$

3) $t \in (a, b)$

$$x(t) = \sum_k a_k \phi_k(t)$$

$$\int_a^b x(t)\,\phi_m^*(t)\,dt = \int_a^b \phi_m^*(t) \sum_k a_k \phi_k(t)\,dt$$

$$= \sum_k a_k \int_a^b \phi_m^*(t)\,\phi_k(t)\,dt \qquad \text{Note: } \int_a^b \phi_m^*(t)\,\phi_k(t)\,dt = \begin{cases} 1 & m = k \\ 0 & m \neq k \end{cases}$$

$$= a_m$$

Thus $a_m = \int_a^b x(t)\,\phi_m^*(t)\,dt$

4) $x(t) = \frac{1}{\sqrt{T}} \sum_k a_k e^{jk\omega_0 t}$

$$a_k = \frac{1}{\sqrt{T}} \int_a^b x(t)\,e^{-jk\omega_0 t}$$

$$= \frac{1}{\sqrt{T}} \int_0^2 t\,e^{-jk\omega_0 t}\,dt \qquad = \frac{1}{\sqrt{T}}\left[\frac{-t}{jk\omega_0} e^{-jk\omega_0 t}\Big|_0^2 + \int_0^2 \frac{1}{jk\omega_0} e^{-jk\omega_0 t}\,dt\right]$$

$$u = t \qquad dv = e^{-jk\omega_0 t}\,dt$$

$$du = dt \qquad v = \frac{-1}{jk\omega_0} e^{-jk\omega_0 t}$$

$$= \frac{1}{\sqrt{2}}\left[\frac{-2}{jk\pi} e^{-jk2\pi} - 0 + \frac{-1}{(jk\omega_0)^2} e^{-jk\omega_0 t}\Big|_0^2\right] \qquad \text{Note: } e^{-jk2\pi} = 1$$

$$= \frac{1}{\sqrt{2}}\left[\frac{-2}{jk\pi} + \frac{1}{k^2\pi^2}\left(e^{-jk2\pi} - 1\right)\right] = \frac{-\sqrt{2}}{jk\pi}$$

5) $x(t) = \sum_k a_k \, \phi_k(t)$ where $\{\phi_k(t)\}$ are orthonormal over $t \in (a, b)$

$$\int_a^b |x(t)|^2 \, dt = \int_a^b x(t) \, x^*(t) \, dt$$

$$= \int_a^b \left(\sum_k a_k \phi_k(t) \right) \left(\sum_\ell a_\ell \phi_\ell(t) \right)^* dt$$

$$= \sum_k \sum_\ell a_k a_\ell^* \int_a^b \phi_k(t) \, \phi_\ell^*(t) \, dt \qquad \int_a^b \phi_k(t) \, \phi_\ell^*(t) dt = \begin{cases} 1 & k = \ell \\ 0 & k \neq \ell \end{cases}$$

$$= \sum_k a_k a_k^*$$

$$= \sum_k |a_k|^2$$