ECE 301 HW4 Solutions

1) a: $y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

$$\frac{d}{dt} y(t) = \frac{d}{dt} \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau) \frac{d}{dt} x(t-\tau)\, d\tau$$

$$= h(t) * \frac{dx(t)}{dt}$$

b: $y[n] = \sum_k x[k]\, h[n-k]$

$$\sum_n y[n] = \sum_n \sum_k x[k]\, h[n-k] = \sum_k \sum_n x[k]\, h[n-k]$$

$$= \sum_k \left[x[k] \sum_{n=-\infty}^{\infty} h[n-k] \right]$$

$$= \left(\sum_k x[k] \right) \left(\sum_n h[n] \right) = \left(\sum_n x[n] \right) \left(\sum_n h[n] \right)$$

c: $x(t)$ is periodic $\iff x(t+kT) = x(t) \qquad k \in \mathbb{Z}$

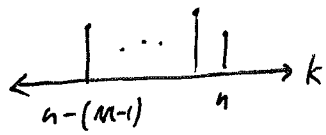
$y(t+kT) = x(t+kT) * h(t)$ since the system is time-invariant

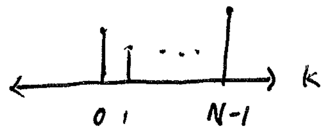
$$= x(t) * h(t)$$

$$= y(t) \iff y(t) \text{ is periodic}$$

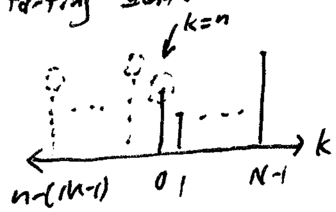
2) $x[n] : 0 \le n < M$ $\qquad$ $h[n] : 0 \le n < N$

a: $x[n-k]$ $\qquad$ $h[k]$ $\qquad$ $y[n] = \sum_k h[k]\, x[n-k]$



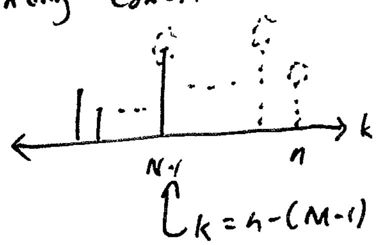


Starting ~~Intermediate~~ Condition:



$n \ge 0$

Ending Condition:



$n-(M-1) \le N-1$

$n \le N+M-2$

$L_1 = 0 \qquad L_2 = N+M-2$

b: $x[n] = u[n] - u[n-5]$ $\qquad$ $h[n] = 2(u[n] - u[n-3])$

$M = 5 \qquad N = 3$

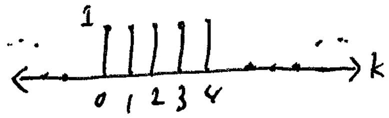
$$\begin{aligned}
y[n] &= x[n] * h[n] \\
&= x[n] * 2(\delta[n] + \delta[n-1] + \delta[n-2]) \\
&= 2x[n] + 2x[n-1] + 2x[n-2] \\
&= 2(u[n] - u[n-5]) + 2(u[n-1] - u[n-6]) + 2(u[n-2] - u[n-7])
\end{aligned}$$

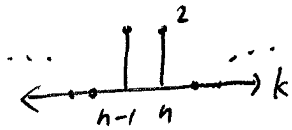
$y[n] \ne 0$ for $0 \le n \le 6$ $\quad$ confirmed!

$N+M-2 = 5+3-2 = 6$

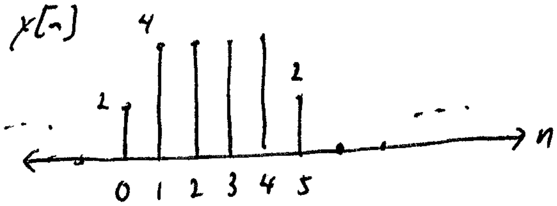
c: $x[n] = u[n] - u[n-5]$ $\qquad h[n] = 2(u[n] - u[n-2])$

$x[k]$ $\qquad M=5$ $\qquad h[n-k]$ $\qquad N=2$





$y[n]$



$N+M-2 = 5+2-2 = 5$

confirmed!

3) a: $x[n] * y[n] = \sum_k x[k]\, y[n-k] = \sum_\ell x[n-\ell]\, y[\ell] = y[n] * x[n]$

Let $\ell = n-k$

b: $(x[n] * y[n]) = w[n] = \sum_k x[k]\, y[n-k]$

$(x[n] * y[n]) * z[n] = \sum_\ell w[\ell]\, z[n-\ell] = \sum_\ell \sum_k x[k]\, y[\ell-k]\, z[n-\ell]$

$= \sum_k \sum_\ell x[k]\, y[\ell-k]\, z[n-\ell]$

$= \sum_k x[k] \sum_\ell y[\ell-k]\, z[n-\ell]$

Let $m = n-\ell$

$\ell = n-m$

$= \sum_k x[k] \sum_m y[n-m-k]\, z[m]$ $\qquad$ Let $q[n] = \sum_m y[n-m]\, z[m]$

$= \sum_k x[k]\, q[n-k]$

$= x[n] * q[n] = x[n] * (y[n] * z[n])$

c: $x[n] * (y[n] + z[n]) = \sum_k x[k]\big(y[n-k] + z[n-k]\big)$

$= \sum_k x[k]\,y[n-k] + \sum_k x[k]\,z[n-k]$

$= x[n] * y[n] + x[n] * z[n]$

d: Proof by contradiction ($\Leftarrow$)

$\boxed{(\Rightarrow)\ \sum_k h[k]\,x[n-k] = \sum_{k=0}^{n} h[k]\,x[n-k] \text{ which is causal}}$

Suppose $h[n_0] \neq 0$ for $n_0 < 0$ & the system is causal

$y[n] = x[n] * h[n] = \sum_k h[k]\,x[n-k]$

$= \sum_{k \neq n_0} h[k]\,x[n-k] + h[n_0]\,x[n-n_0]$

Note $y[n]$ is a function of $x[n-n_0]$, which is a future value of $x[n]$ since $n_0 < 0$. Thus, there is a contradiction and $h[n] = 0$ for all $n < 0 \Leftrightarrow$ causal system

4) a: $h[n] = \left(\frac{1}{2}\right)^n u[-n]$

non-causal : $h[-1] = 2 \neq 0$

~~warm~~ unstable : $\sum_n |h[n]| = \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} 2^n = \infty$

b: ~~ca~~ $h[n] = \left(-\frac{1}{2}\right)^n u[n] + (1.01)^n u[n-1]$

causal : $h[n] = 0$ for all $n < 0$

unstable : $\sum_{n=1}^{\infty} (1.01)^n = \infty$

c: $h(t) = e^{2t} u(-1-t)$

non-causal : $h(-2) = e^{-4} \neq 0$

unstable : $\int_{-\infty}^{\infty} |h(t)|\,dt = \int_{-\infty}^{-1} e^{2t}\,dt = \frac{1}{2} e^{2t}\Big|_{-\infty}^{-1} = \frac{1}{2}\left(e^{-2} - 0\right) < \infty$

d: $h(t) = t e^{-t} u(t)$

causal : $h(t) = 0$ for all $t < 0$

stable : $\int_{-\infty}^{\infty} |h(t)|\,dt = \int_0^{\infty} t e^{-t}\,dt = -te^{-t}\Big|_0^{\infty} + \int_0^{\infty} e^{-t}\,dt = -0 + 0 - e^{-t}\Big|_0^{\infty} = 1 < \infty$

$u = t \quad dv = e^{-t}dt$

$du = dt \quad v = -e^{-t}$

5) $\frac{d}{dt} y(t) = -a y(t) + x(t)$

$$h(t) = e^{-at} u(t)$$

$$\frac{d}{dt} h(t) = -a e^{-at} u(t) + e^{-at} \overset{\delta(t)}{\cancel{u(t)}} = -a e^{-at} u(t) + \delta(t)$$

$$= -a y(t) + \delta(t)$$

6) $y[n] = \frac{1}{2} y[n-1] + x[n]$

a: $h[n] = \left(\frac{1}{2}\right)^n u[n]$

b: $y[n] = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k x[n-k]$

c: $y[n] = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k u[n-k] = u[n] \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k$

$$= u[n] \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = u[n]\, 2\left(1 - \left(\tfrac{1}{2}\right)^{n+1}\right)$$

$$= u[n]\left(2 - \left(\tfrac{1}{2}\right)^n\right)$$

d: $y[n] = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k (1) = \frac{1}{1-\frac{1}{2}} = 2$

7) $S[e^{j\omega t}] = c\, e^{j(\omega t + \phi)}$ for an LTI system

a: NOT LTI

b: perhaps LTI

c: NOT LTI

d: NOT LTI

e: NOT LTI

f: perhaps LTI

8) a: $y[n] = \sum_k h[k]\, x[n-k]$

b: $x[n] = e^{j\omega n}$

$$y[n] = \sum_k h[k]\, e^{j\omega(n-k)} = e^{j\omega n} \underbrace{\sum_k h[k] e^{-j\omega k}}_{C(\omega) = \sum_k h[k] e^{-j\omega k}}$$

c: $C^*(\omega) = \sum_k h^*[k] e^{j\omega k} = \cancel{h}\sum_k h[k]\, e^{j\omega k}$ if $h[k]$ is real-valued

$= C(-\omega)$

d: $x[n] = \cos(\omega n) = \frac{1}{2}(e^{j\omega n} + e^{-j\omega n})$

$$y[n] = \tfrac{1}{2} C(\omega) e^{j\omega n} + \tfrac{1}{2} C(-\omega) e^{-j\omega n} = \tfrac{1}{2} C(\omega) e^{j\omega n} + \tfrac{1}{2} C^*(\omega) e^{-j\omega n}$$

Let $C(\omega) = a + jb$

$$= \tfrac{1}{2}(a+jb)(\cos(\omega n) + j\sin(\omega n)) + \tfrac{1}{2}(a-jb)(\cos(\omega n) - j\sin(\omega n))$$

$$= \tfrac{1}{2}\cos(\omega n)(a+jb+a-jb) + \tfrac{j}{2}\sin(\omega n)(a+jb-a+jb)$$

$$= \mathrm{Re}\{C(\omega)\}\cos(\omega n) - \mathrm{Im}\{C(\omega)\sin(\omega n)\}$$

e: $x[n] = B\cos(\omega n + \phi) = \frac{B}{2}\left(e^{j(\omega n+\phi)} + e^{-j(\omega n+\phi)}\right)$

$$y[n] = \tfrac{B}{2} e^{j\phi} C(\omega) e^{j\omega n} + \tfrac{B}{2} e^{-j\phi} C(-\omega) e^{-j\omega n}$$

$$= \tfrac{B}{2} e^{j\phi} C(\omega) e^{j\omega n} + \tfrac{B}{2} e^{-j\phi} C^*(\omega) e^{-j\omega n}$$

$$= \tfrac{B}{2} \cancel{e^{j\phi}} (a+jb)(\cos(\omega n+\phi) + j\sin(\omega n+\phi)) + \tfrac{\cancel{j}B}{2}(a-jb)(\cos(\omega n+\phi) - j\sin(\omega n+\phi))$$

$$= \tfrac{B}{2}\cos(\omega n+\phi)(a+jb+a-jb) + \tfrac{jB}{2}\sin(\omega n+\phi)(a+jb-a+jb)$$

$$= B\,\mathrm{Re}\{C(\omega)\}\cos(\omega n+\phi) - B\,\mathrm{Im}\{C(\omega)\}\sin(\omega n)$$

$f: \; x[n] = \sin(\omega n) = \frac{1}{2j}\left(e^{j\omega n} - e^{-j\omega n}\right)$

$$
\begin{aligned}
y[n] &= \frac{1}{2j} C(\omega) e^{j\omega n} - \frac{1}{2j} C^*(\omega) e^{-j\omega n} \\
&= \frac{1}{2j}(a+jb)(\cos(\omega n) + j\sin(\omega n)) - \frac{1}{2j}(a-jb)(\cos(\omega n) - j\sin(\omega n)) \\
&= \frac{1}{2j}\cos(\omega n)(a+jb-a+jb) + \frac{1}{2}\sin(\omega n)(a+jb-a+jb) \\
&= \operatorname{Im}\{C(\omega)\}\sin(\omega n) - \operatorname{Re}\{C(\omega)\}\cos(\omega n)
\end{aligned}
$$

$g: \; x[n] = B\sin(\omega n + \phi) = \frac{B}{2j}\left(e^{j(\omega n+\phi)} - e^{-j(\omega n+\phi)}\right)$

$$
\begin{aligned}
y[n] &= \frac{B}{2j} e^{j\phi} C(\omega) e^{j\omega n} - \frac{B}{2j} e^{-j\phi} C^*(\omega) e^{-j\omega n} \\
&= \frac{B}{2j}(a+jb)(\cos(\omega n+\phi) + j\sin(\omega n+\phi)) - \frac{B}{2j}(a-jb)(\cos(\omega n+\phi) - j\sin(\omega n+\phi)) \\
&= \frac{B}{2j}\cos(\omega n+\phi)(a+jb-a+jb) + \frac{B}{2}\sin(\omega n+\phi)(a+jb+a\mp jb) \\
&= B\operatorname{Im}\{C(\omega)\}\sin(\omega n+\phi) \mp B\operatorname{Re}\{C(\omega)\}\cos(\omega n+\phi)
\end{aligned}
$$