ECE 301 HW3 Solutions

1) a: $y[n] = \sum_{k=0}^{\infty} b_k \, x[n-k]$

i. Let $y_1[n] = \sum_{k=0}^{\infty} b_k \, x_1[n-k]$ & $y_2[n] = \sum_{k=0}^{\infty} b_k \, x_2[n-k]$

$$x_3[n] = a_1 x_1[n] + a_2 x_2[n]$$

$$y_3[n] = \sum_{k=0}^{\infty} b_k x_3[n-k] = \sum_{k=0}^{\infty} b_k \left(a_1 x_1[n-k] + a_2 x_2[n-k] \right)$$

$$= a_1 \sum_{k=0}^{\infty} b_k x_1[n-k] + a_2 \sum_{k=0}^{\infty} b_k x_2[n-k] = a_1 y_1[n] + a_2 y_2[n]$$

Therefore, the system is linear

ii. Let $y_1[n]$ be output of shifted input, $x[n-n_0]$
$y_2[n]$ be shifted output of $x[n]$, $y[n-n_0]$

$$\left.\begin{aligned} y_1[n] &= \sum_{k=0}^{\infty} b_k \, x[n-k-n_0] \\ y_2[n] &= \sum_{k=0}^{\infty} b_k \, x[n-k-n_0] \end{aligned}\right\} \quad y_1[n] = y_2[n] \Rightarrow \text{The system is time-invariant}$$

iii. $h[n] = \sum_{k=0}^{\infty} b_k \, \delta[n-k] = \begin{cases} b_n & n \geq 0 \\ 0 & n < 0 \end{cases} = b_n \, u[n]$

b: $y[n] = \frac{1}{3} x[n] - \frac{1}{6} x[n+1] - \frac{1}{6} x[n-1]$

i. Let $y_1[n] = \frac{1}{3} x_1[n] - \frac{1}{6} x_1[n+1] - \frac{1}{6} x_1[n-1]$

$$y_2[n] = \frac{1}{3} x_2[n] - \frac{1}{6} x_2[n+1] - \frac{1}{6} x_2[n-1]$$

$$x_3[n] = a_1 x_1[n] + a_2 x_2[n]$$

$$y_3[n] = \frac{1}{3} x_3[n] - \frac{1}{6} x_3[n+1] - \frac{1}{6} x_3[n-1]$$

$$= \frac{1}{3}\left(a_1 x_1[n] + a_2 x_2[n] \right) - \frac{1}{6}\left(a_1 x_1[n+1] + a_2 x_2[n+1] \right) - \frac{1}{6}\left(a_1 x_1[n-1] + a_2 x_2[n-1] \right)$$

$$= a_1 \left(\frac{1}{3} x_1[n] - \frac{1}{6} x_1[n+1] - \frac{1}{6} x_1[n-1] \right) + a_2 \left(\frac{1}{3} x_2[n] - \frac{1}{6} x_2[n+1] - \frac{1}{6} x_2[n-1] \right)$$

$$= a_1 y_1[n] + a_2 y_2[n]$$

Therefore, the system is linear

ii. $y_1[n] = \frac{1}{3}x[n-n_0] - \frac{1}{6}x[n-n_0+1] - \frac{1}{6}x[n-n_0-1]$

$y_2[n] = \frac{1}{3}x[n-n_0] - \frac{1}{6}x[n-n_0+1] - \frac{1}{6}x[n-n_0-1]$

$y_1[n] = y_2[n] \Rightarrow$ The system is time-invariant

iii. $h[n] = \frac{1}{3}\delta[n] - \frac{1}{6}\delta[n+1] - \delta[n-1]$

c: $y[n] = \frac{1}{2}y[n-1] + x[n]$

i. Let $y_1[n] = \frac{1}{2}y_1[n-1] + x_1[n]$ & $y_2[n] = \frac{1}{2}y_2[n-1] + x_2[n]$

$x_3[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$

$y_3[n] = \frac{1}{2}y_3[n-1] + x_3[n]$

$= \frac{1}{2}y_3[n-1] + \alpha_1 x_1[n] + \alpha_2 x_2[n]$

$y_1[n] - \frac{1}{2}y_1[n-1] = x_1[n]$ $\qquad$ $y_2[n] - \frac{1}{2}y_2[n] = x_2[n]$

$y_3[n] - \frac{1}{2}y_3[n-1] = \alpha_1 x_1[n] + \alpha_2 x_2[n] \Rightarrow$ The system is linear

ii. $\left.\begin{array}{l} y_1[n] - \frac{1}{2}y_1[n-1] = x_1[n-n_0] \\ y_2[n] - \frac{1}{2}y_2[n-1] = x[n-n_0] \end{array}\right\} \Rightarrow$ The system is time-invariant

iii. $h[n] = \begin{cases} 1 & n=0 \\ \frac{1}{2} & n=1 \\ \frac{1}{4} & n=2 \\ \vdots & \end{cases}$ $\qquad h[n] = \left(\frac{1}{2}\right)^n \mathcal{U}[n]$

2) $y[n] = x[n] - 3x[n-1] + 2x[n-2]$

a: $h[n] = \delta[n] - 3\delta[n-1] + 2\delta[n-2]$

b: $y[n] = [\delta[n] - 3\delta[n-1] + 2\delta[n-2]] * x[n]$

c: $y[n] = u[n] - 3u[n-1] + 2u[n-2]$

$$= \begin{cases} 1 & n=0 \\ -2 & n=1 \\ 0 & ow \end{cases} \qquad = \delta[n] - 2\delta[n-1]$$

d: $y[n] = 1 - 3(1) + 2(1) = 0$

3) a: $y(t) = \int_{-\infty}^{\infty} r(\tau - t)\, x(\tau)\, d\tau$

i. Let $y_1(t) = \int_{-\infty}^{\infty} r(\tau - t)\, x_1(\tau)\, d\tau$ & $y_2(t) = \int_{-\infty}^{\infty} r(\tau - t)\, x_2(\tau)\, d\tau$

$x_3(t) = a_1 x_1(t) + a_2 x_2(t)$

$$y_3(t) = \int_{-\infty}^{\infty} r(\tau - t)\big(a_1 x_1(\tau) + a_2 x_2(\tau)\big)\, d\tau$$

$$= a_1 \int_{-\infty}^{\infty} r(\tau - t)\, x_1(\tau)\, d\tau + a_2 \int_{-\infty}^{\infty} r(\tau - t)\, x_2(\tau)\, d\tau$$

$= a_1 y_1(t) + a_2 y_2(t) \Rightarrow$ the system is linear

ii. Let $y_1(t)$ be the output of the shifted input, ~~x(t)~~ $x(t-t_0)$

$y_2(t)$ be the shifted output of $x(t)$, $y(t-t_0)$

$$y_1(t) = \int_{-\infty}^{\infty} r(\tau - t)\, x_1(\tau - t_0)\, d\tau$$

$$y_2(t) = \int_{-\infty}^{\infty} r(\tau - (t - t_0))\, x(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} r(\tau - t)\, x(\tau - t_0)\, d\tau$$

$y_1(t) = y_2(t) \Rightarrow$ The system is linear

iii. $h(t) = \int_{-\infty}^{\infty} r(\tau - t)\, \delta(\tau)\, d\tau =$

$= r(-t)$

b: $y(t) = x(t) + 2x(t+1) + 3x(t-1)$

i. Let $y_1(t) = x_1(t) + 2x_1(t+1) + 3x_1(t-1)$

$y_2(t) = x_2(t) + 2x_2(t+1) + 3x_2(t-1)$

$x_3(t) = a_1 x_1(t) + a_2 x_2(t)$

$$y_3(t) = x_3(t) + 2x_3(t+1) + 3x_3(t-1)$$
$$= a_1 x_1(t) + a_2 x_2(t) + 2(a_1 x_1(t+1) + a_2 x_2(t+1)) + 3(a_1 x_1(t-1) + a_2 x_2(t-1))$$
$$= a_1(x_1(t) + 2x_1(t+1) + 3x_1(t-1)) + a_2(x_2(t) + 2x_2(t+1) + 3x_2(t-1))$$
$$= a_1 y_1(t) + a_2 y_2(t)$$

Therefore, the system is linear

ii. $y_1(t) = x(t-t_0) + 2x(t-t_0+1) + 3x(t-t_0-1)$

$y_2(t) = x(t-t_0) + 2x(t-t_0+1) + 3x(t-t_0-1)$

Therefore, the system is time-invariant

iii. $h(t) = \delta(t) + 2\delta(t+1) + 3\delta(t-1)$

c: $\frac{d}{dt} y(t) = -x(t) \Rightarrow y(t) = -\int_{-\infty}^{t} x(\tau)d\tau$

i. Let $y_1(t) = -\int_{-\infty}^{t} x_1(\tau)d\tau$ & $y_2(t) = -\int_{-\infty}^{t} x_2(\tau)d\tau$

$x_3(t) = a_1 x_1(t) + a_2 x_2(t)$

$$y_3(t) = -\int_{-\infty}^{t} (a_1 x_1(\tau) + a_2 x_2(\tau))d\tau = a_1\left[-\int_{-\infty}^{t} x_1(\tau)d\tau\right] + a_2\left[-\int_{-\infty}^{t} x_2(\tau)d\tau\right]$$
$$= a_1 y_1(t) + a_2 y_2(t)$$

Therefore the system is linear

ii. $y_1(t) = -\int_{-\infty}^{t} x_n(\tau - t_0)d\tau$

$= -\int_{-\infty}^{t-t_0} x(\tau)d\tau$ $\qquad$ $y_1(t) = y_2(t) \Rightarrow$ The system is time-invariant

$y_2(t) = -\int_{-\infty}^{t-t_0} x(\tau)d\tau$

iii. $h(t) = -\int_{-\infty}^{t} \delta(\tau)d\tau = \begin{cases} -1 & t > 0 \\ 0 & t < 0 \end{cases}$

$h(t) = -u(t)$

4) Since it is time invariant,

$$T[\delta_{n-3}] = \delta_{n+1} + 2\delta_n + \delta_{n-1} \Rightarrow T[\delta_n] = \delta_{n+4} + 2\delta_{n+3} + \delta_{n+2}$$

a: If the system is linear, $T[\delta_n + 2\delta_{n-2}] = T[\delta_n] + 2T[\delta_{n-2}]$

$$T[3\delta_{n-2}] = 3T[\delta_{n-2}]$$

$$T[\delta_n] + 2T[\delta_{n-2}] = \delta_{n+4} + 2\delta_{n+3} + \delta_{n+2} + 2(\delta_{n+2} + 2\delta_{n+1} + \delta_n)$$

$$= \delta_{n+4} + 2\delta_{n+3} + 3\delta_{n+2} + 4\delta_{n+1} + 2\delta_n \neq T[\delta_n + 2\delta_{n-2}]$$

$$3T[\delta_{n-2}] = 3(\delta_{n+2} + 2\delta_{n+1} + \delta_n) \neq T[3\delta_{n-2}]$$

Therefore, the system is non-linear

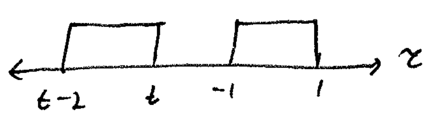
b: $T[\delta_n] = \delta_{n+4} + 2\delta_{n+3} + \delta_{n+2}$

5) a: $h(t) = u(t+1) - u(t-1) = \begin{cases} 1 & -1<t<1 \\ 0 & \text{ow} \end{cases}$ $\qquad x(t) = u(t) - u(t-2) = \begin{cases} 1 & 0<t<2 \\ 0 & \text{ow} \end{cases}$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

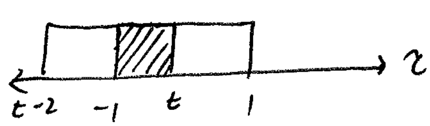
case 1: $t<-1$

$y(t) = 0$



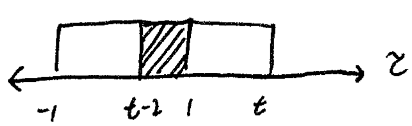
case 2: $-1<t<1$

$$y(t) = \int_{-1}^{t} (1)\, d\tau = \tau \Big|_{-1}^{t} = t+1$$



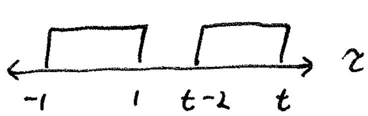
case 3: $1<t<3$

$$y(t) = \int_{t-2}^{1} (1)\, d\tau = \tau \Big|_{t-2}^{1} = 3-t$$

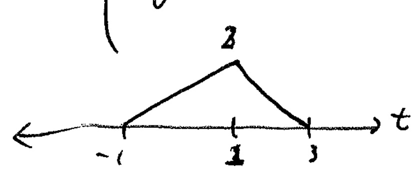


case 4: $t>3$

$y(t) = 0$



$$y(t) = \begin{cases} t+1 & -1<t<1 \\ 3-t & 1<t<3 \\ 0 & \text{ow} \end{cases}$$



5) b: $h(t) = e^{-\alpha t}\,\mathcal{U}(t)$ $\quad x(t) = \mathcal{U}(t)$ $\quad \alpha > 0$

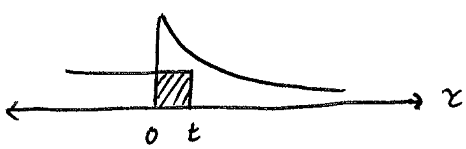
Case 1:

$t < 0$

$y(t) = 0$

Case 2:

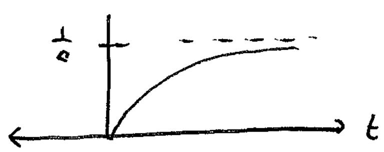
$t > 0$



$$y(t) = \int_0^t (1)\, e^{-\alpha\tau}\, d\tau$$

$$= -\frac{1}{\alpha} e^{-\alpha\tau}\Big|_0^t = -\frac{1}{\alpha}\left(e^{-\alpha t} - 1\right)$$

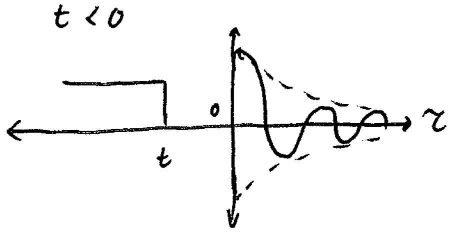
$$y(t) = \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right)\mathcal{U}(t)$$



c: $h(t) = e^{-\alpha t}\cos(\omega t)\,\mathcal{U}(t)$ $\quad x(t) = \mathcal{U}(t)$ $\quad \alpha > 0$

Case 1:

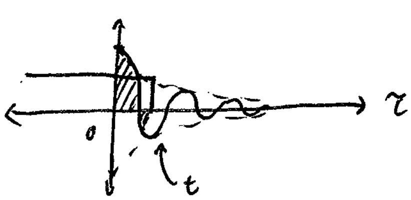
$t < 0$



$y(t) = 0$

Case 2:

$t > 0$



$$y(t) = \int_0^t e^{-\alpha\tau}\cos(\omega\tau)\, d\tau$$

$$= \frac{1}{2}\int_0^t e^{-\alpha\tau}\left(e^{j\omega\tau} + e^{-j\omega\tau}\right) d\tau$$

$$= \frac{1}{2}\left[\frac{1}{j\omega - \alpha} e^{\tau(j\omega - \alpha)} + \frac{1}{-\alpha - j\omega} e^{-\tau(\alpha + j\omega)}\right]_0^t$$

$$y(t) = \frac{1}{2}\left[\frac{-\alpha - j\omega + j\omega - \alpha}{-j\alpha\omega + \omega^2 + \alpha^2 + j\alpha\omega} e^{-\alpha\tau}\left(e^{j\omega\tau} + e^{-j\omega\tau}\right)\right]_0^t$$

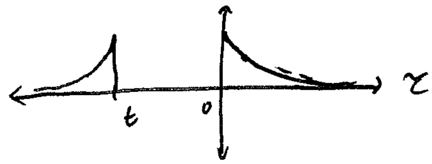
$$= \frac{-2\alpha}{\alpha^2 + \omega^2} e^{-\alpha\tau}\cos(\omega\tau)\Big|_0^t = \frac{2\alpha}{\alpha^2 + \omega^2}\left(1 - e^{-\alpha t}\cos(\omega t)\right)$$

$$y(t) = \frac{2\alpha}{\alpha^2 + \omega^2}\left(1 - e^{-\alpha t}\cos(\omega t)\right)\mathcal{U}(t)$$

5) d: $h(t) = e^{-at}\,\mathcal{U}(t)$ $\quad x(t) = e^{-bt}\,\mathcal{U}(t)$ $\quad a = b > 0$

Case 1:

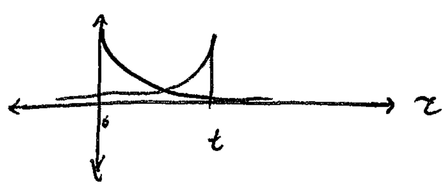
$t < 0$



$y(t) = 0$

Case 2:

$t > 0$



$$y(t) = \int_0^t e^{-a\tau} e^{-b(t-\tau)}\, d\tau$$

$$= e^{-bt} \int_0^t e^{-\tau(a-b)}\, d\tau$$

$$= e^{-bt} \left[-\frac{1}{a-b} e^{-\tau(a-b)} \right]_0^t$$

$$= e^{-bt} \left(\frac{1}{a-b}\right)\left(1 - e^{-t(a-b)}\right)$$

$$y(t) = \frac{1}{a-b}\left(e^{-bt} - e^{-at}\right) \mathcal{U}(t)$$

6) a: $h[n] = a^n u[n]$ $\quad x[n] = b^n u[n]$ $\quad a \neq b$

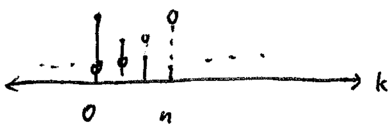
Case 1:

$n < 0$

$$y[n] = \sum_k h[k]\, x[n-k] = 0$$

Case 2:

$n \geq 0$



$$y[n] = \sum_{k=0}^{n} a^k b^{n-k} = b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k$$

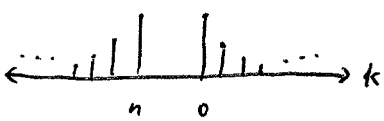
$$= b^n \frac{1-\left(\frac{a}{b}\right)^{n+1}}{1-\frac{a}{b}} = \frac{b^{n+1}-a^{n+1}}{b-a}$$

$$y[n] = \frac{b^{n+1}-a^{n+1}}{b-a}\, u[n]$$

b: $h[n] = a^n u[n]$ $\quad x[n] = a^n u[n]$

Case 1:

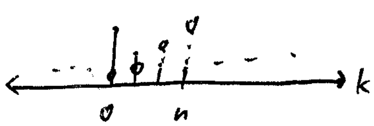
$n < 0$



$$y[n] = 0$$

Case 2:

$n \geq 0$



$$y[n] = \sum_{k=0}^{n} a^k a^{n-k} = a^n \sum_{k=0}^{n} (1)$$

$$= (n+1)\, a^n$$

$$y[n] = (n+1)\, a^n u[n]$$

6) c: $h[n] = \alpha^n u[n] \qquad x[n] = \cos(\omega n) \qquad |\alpha| < 1$

$$y[n] = \sum_k h[k]\, x[n-k]$$

$$= \sum_{k=0}^{\infty} \alpha^k \cos(\omega(n-k)) = \frac{1}{2}\sum_{k=0}^{\infty} \alpha^k \left(e^{j\omega(n-k)} + e^{-j\omega(n-k)}\right)$$

$$= \frac{1}{2}\sum_{k=0}^{\infty} \left(\alpha e^{-j\omega}\right)^k e^{j\omega n} + \frac{1}{2}\sum_{k=0}^{\infty} \left(\alpha e^{j\omega}\right)^k e^{-j\omega n}$$

$$= \frac{1}{2} e^{j\omega n} \frac{1}{1-\alpha e^{-j\omega}} + \frac{1}{2} e^{-j\omega n} \frac{1}{1-\alpha e^{j\omega}} \qquad \text{since } |\alpha| < 1$$

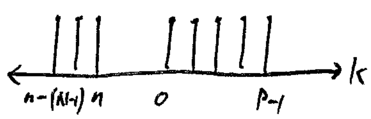
$$= \frac{1}{2} \frac{1-\alpha e^{-j\omega} + 1 - \alpha e^{j\omega}}{1-\alpha e^{j\omega} - \alpha e^{-j\omega} + \alpha^2} \left(e^{j\omega n} + e^{-j\omega n}\right)$$

$$= \frac{2 - 2\alpha\cos(\omega)}{1 - 2\alpha\cos(\omega) + \alpha^2} \cos(\omega n)$$

d: $h[n] = u[n] - u[n-N] \qquad x[n] = u[n] - u[n-P] \qquad P > N$

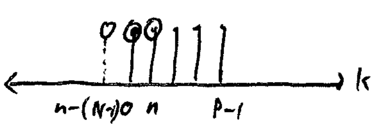
$$h[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{ow} \end{cases} \qquad x[n] = \begin{cases} 1 & 0 \le n \le P-1 \\ 0 & \text{ow} \end{cases}$$

Case 1: $n < 0$



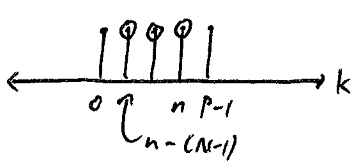
$y[n] = 0$

Case 2: $0 \le n < N$



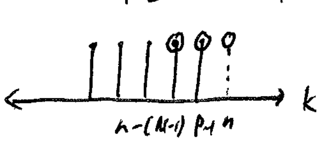
$$y[n] = \sum_{k=0}^{n} (1) = n+1$$

Case 3: $N \le n < P$



$$y[n] = \sum_{k=n-(N-1)}^{n} (1) = N-1+1 = N$$

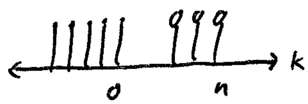
Case 4: $P \le n < N+P$



$$y[n] = \sum_{k=n-(N-1)}^{P-1} (1) = P-1-(n-(N-1))+1$$

$$= P - n + N - 1 = (P+N-1) - n$$

case 5: $n \geq P+N$



$y[n] = 0$

$$y[n] = \begin{cases} n+1 & 0 \leq n < N \\ N & N \leq n < P \\ (P+N-1)-n & P \leq n < N+P \\ 0 & n \geq N+P \end{cases}$$