ECE 301 HW1 Solutions

1) a: i. $z = 1 + j\sqrt{3}$

$|z| = \sqrt{1+3} = 2 \qquad \angle z = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 0.7137 \text{ rad}$

$z = 2e^{j0.7137}$

ii. $z = a + jb\,, \quad a > 0$

$|z| = \sqrt{a^2+b^2} \qquad \angle z = \tan^{-1}\left(\frac{b}{a}\right)$

$z = \sqrt{a^2+b^2}\, e^{j\tan^{-1}\left(\frac{b}{a}\right)}$

iii. $z = (a+jb)^3\,, \quad a > 0$

Let $z_0 = a + jb$, $\quad z = z_0^3 \qquad |z_0| = \sqrt{a^2+b^2} \qquad \angle z_0 = \tan^{-1}\left(\frac{b}{a}\right)$

$z = (a^2+b^2)^{3/2}\left(e^{j\tan^{-1}\left(\frac{b}{a}\right)}\right)^3 = (a^2+b^2)^{3/2} e^{j3\tan^{-1}\left(\frac{b}{a}\right)}$

iv. $z = \dfrac{e^{j\psi}}{e^{j\phi}} = e^{j\psi}e^{-j\phi} = e^{j(\psi-\phi)}$

V. $z = \dfrac{e^{j\frac{\pi}{3}} - 1}{1 + j\sqrt{3}} = \dfrac{z_1}{z_2}$

$z_1 = \cos\left(\frac{\pi}{3}\right) - 1 + j\sin\left(\frac{\pi}{3}\right) = -\frac{1}{2} + j\frac{\sqrt{3}}{2} = (1)e^{j\frac{2\pi}{3}}$

$z_2 = 1 + j\sqrt{3} = 2e^{j\frac{\pi}{3}}$

$z = \dfrac{z_1}{z_2} = \dfrac{e^{j\frac{2\pi}{3}}}{2e^{j\frac{\pi}{3}}} = \frac{1}{2}e^{j\left(\frac{2\pi}{3} - \frac{\pi}{3}\right)} = \frac{1}{2}e^{j\frac{\pi}{3}}$

1) b: i. $z = e^{j\theta} = \cos\theta + j\sin\theta$

ii. $z = e^{a+j\theta} = e^{a}e^{j\theta} = e^{a}(\cos\theta + j\sin\theta)$

$= e^{a}\cos\theta + je^{a}\sin\theta$

iii. $z = \dfrac{e^{j\psi}}{e^{j\phi}} = e^{j(\psi-\phi)} = \cos(\psi-\phi) + j\sin(\psi-\phi)$

iv. $z = \dfrac{1}{c+jd} \cdot \dfrac{c-jd}{c-jd} = \dfrac{c-jd}{c^2+d^2} = \dfrac{c}{c^2+d^2} - j\dfrac{d}{c^2+d^2}$

v. $z = \dfrac{e+jf}{c+jd} \cdot \dfrac{c-jd}{c-jd} = \dfrac{ce - jde + jcf + fd}{c^2+d^2} = \dfrac{ce+fd}{c^2+d^2} + j\dfrac{cf-de}{c^2+d^2}$

2) $Ae^{j\theta} = a + jb$

$e^{j\theta} = \dfrac{a}{A} + j\dfrac{b}{A}$

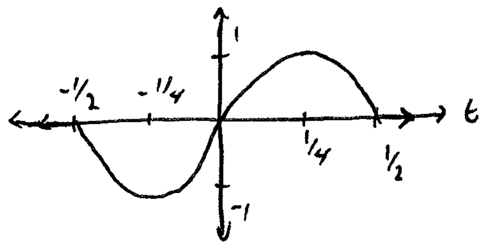
$$\theta = \begin{cases} \tan^{-1}\left(\frac{b}{a}\right) & a > 0 \\ \pi + \tan^{-1}\left(\frac{b}{a}\right) & a < 0 \\ \frac{\pi}{2} & a = 0,\ b > 0 \\ \frac{3\pi}{2} & a = 0,\ b < 0 \end{cases}$$

3) $x(t) = \sin(2\pi t)\, u(t+\tfrac{1}{2})\, u(-t+\tfrac{1}{2})$

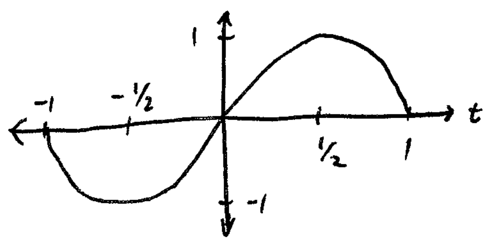
Note: $u(t+\tfrac{1}{2}) = \begin{cases} 1 & t > -\tfrac{1}{2} \\ 0 & \text{ow} \end{cases}$ $\qquad u(-t+\tfrac{1}{2}) = \begin{cases} 1 & t < \tfrac{1}{2} \\ 0 & \text{ow} \end{cases}$

thus, $u(t+\tfrac{1}{2})\, u(-t+\tfrac{1}{2}) = \begin{cases} 1 & -\tfrac{1}{2} < t < \tfrac{1}{2} \\ 0 & \end{cases}$

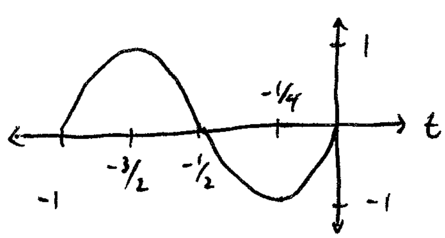
a: $x(t) = \begin{cases} \sin(2\pi t) & -\tfrac{1}{2} < t < \tfrac{1}{2} \\ 0 & \text{ow} \end{cases}$



b: $x(t/2) = \begin{cases} \sin(\pi t) & -1 < t < 1 \\ 0 & \end{cases}$



c: $x(-t-\tfrac{1}{2}) = \begin{cases} \sin(2\pi(-t-\tfrac{1}{2})) & -\tfrac{1}{2} < -t-\tfrac{1}{2} < \tfrac{1}{2} \\ 0 & \text{ow} \end{cases} = \begin{cases} \sin(-2\pi t - \pi) & -1 < t < 0 \\ \text{ow} & \text{ow} \end{cases}$



4) a: i. $x(t) = b\cos(2\pi f t + \theta)$

$x(t+T) = b\cos(2\pi f(t+T)+\theta) = b\cos(2\pi f t + 2\pi f T + \theta)$

for $x(t) = x(t+T)$, $2\pi f T = 2\pi k$, $k \in \mathbb{Z}$

$\Rightarrow T = \frac{k}{f}$

Therefore, $x(t)$ is periodic with fundamental $(k=1)$ period $\frac{1}{f}$

ii. $x(t) = b\cos(\omega t + \theta)$

$x(t+T) = b\cos(\omega t + \omega T + \theta)$

$\omega T = k\,2\pi \Rightarrow T = \frac{2\pi k}{\omega}$

$x(t)$ is periodic with fundamental period $\frac{2\pi}{\omega}$

iii. $x(t) = b\cos(\omega_1 t + \theta) + c\sin(2\omega_1 t + \phi)$

$x(t+T) = b\cos(\omega_1 t + \omega_1 T + \theta) + c\sin(2\omega_1 t + 2\omega_1 T + \phi)$

$\omega_1 T = 2\pi k_1$, $2\omega_1 T = 2\pi k_2$ $k_1, k_2 \in \mathbb{Z}$

$T = \frac{2\pi k_1}{\omega_1} = \frac{\pi k_2}{\omega_1} \Rightarrow k_2 = 2k_1$

$x(t)$ is periodic with fundamental period $\frac{2\pi}{\omega_1}$

iv. $x(t) = b\cos(\omega_1 t + \theta) + c\sin(\sqrt{2}\,\omega_1 t + \phi)$

$x(t+T) = b\cos(\omega_1 t + \omega_1 T + \theta) + c\sin(\sqrt{2}\omega_1 t + \sqrt{2}\omega_1 T + \phi)$

$\omega_1 T = 2\pi k_1$, $\sqrt{2}\omega_1 T = 2\pi k_2$

$T = \frac{2\pi k_1}{\omega_1} = \frac{2\pi k_2}{\sqrt{2}\,\omega_1} \Rightarrow k_2 = \sqrt{2}\,k_1$ which means k_2 must take on non-integer values to make $x(t+T) = x(t)$

Therefore, $x(t)$ is not periodic

4) b: i. $x[k] = b\cos(\pi k + \theta)$

$x[k+N] = b\cos(\pi k + \pi N + \theta)$

$\pi N = 2\pi n \qquad n \in \mathbb{Z}$

$N = 2n \Rightarrow N \in \mathbb{Z}$

Therefore, $x[k]$ is periodic with period 2 samples

ii. $x[k] = b\cos(\frac{\pi}{4}k + \theta)$

$x[k+N] = b\cos(\frac{\pi}{4}k + \frac{\pi}{4}N + \theta)$

$\frac{\pi}{4}N = 2\pi n \Rightarrow N = 8n \in \mathbb{Z}$

$x[k]$ is periodic with fundamental period 8 samples

iii. $x[k] = b\cos(\frac{\pi}{2}k + \theta)$

$x[k+N] = b\cos(\frac{\pi}{2}k + \frac{\pi}{2}N + \theta)$

$\frac{\pi}{2}N = 2\pi n \Rightarrow N = 4n \in \mathbb{Z}$

$x[k]$ is periodic with fundamental period 4 samples

iv. $x[k] = b\cos(\frac{7\pi}{4}k + \theta)$

$x[k+N] = b\cos(\frac{7\pi}{4}k + \frac{7\pi}{4}N + \theta)$

$\frac{7\pi}{4}N = 2\pi n \Rightarrow N = \frac{8}{7}n$

for $n = 7\ell$, $\ell \in \mathbb{Z}$, $N \in \mathbb{Z}$

$x[k]$ is periodic with fundamental period 8 samples

v. $x[k] = b\cos(\frac{12\pi}{11}k + \theta)$

$x[k+N] = b\cos(\frac{12\pi}{11}k + \frac{12\pi}{11}N + \theta)$

$\frac{12\pi}{11}N = 2\pi n \Rightarrow N = \frac{11}{6}n$

for $n = 6\ell$, $N \in \mathbb{Z}$

$x[k]$ is periodic with fundamental period 11 samples

5) a: $x(t) = e^{-t}u(t)$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^{\infty} e^{-2t} dt = -\frac{1}{2} e^{-2t}\Big|_0^{\infty} = \frac{1}{2}$$

$$P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |e^{-t}u(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T}\int_0^T e^{-2t} dt = \lim_{T\to\infty} \frac{1}{2T}\left(-\frac{1}{2}e^{-2t}\right)\Big|_0^T$$

$$= \lim_{T\to\infty} \frac{1}{2T}\left(\frac{1}{2} - \frac{1}{2}e^{-2T}\right) = 0$$

b: $x(t) = \cos(t)$

$$E = \int_{-\infty}^{\infty} \cos^2(t)\, dt = \int_{-\infty}^{\infty} \frac{1}{2}(1+\cos(2t))\, dt = \infty$$

$$P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cos^2(t)\, dt = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \frac{1}{2}(1+\cos(2t))\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\left[\frac{1}{2}t + \frac{1}{4}\sin(2t)\right]_{-T}^{T} = \lim_{T\to\infty} \frac{1}{2T}\left[T + \frac{1}{4}\sin(2T) + \frac{1}{4}\sin(2T)\right]$$

$$= \frac{1}{2} + 0 = \frac{1}{2}$$

c: $x[n] = e^{j(\frac{\pi}{2}n + \frac{\pi}{4})}$

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} \left|e^{j(\frac{\pi}{2}n + \frac{\pi}{4})}\right|^2 = \sum_{n=-\infty}^{\infty} 1 = \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} (1) = 1$$

d: $x[n] = \cos(\frac{\pi}{4}n)$

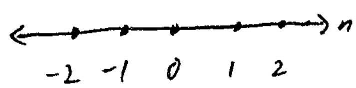
$$E = \sum_{n=-\infty}^{\infty} \cos^2(\tfrac{\pi}{4}n) = \sum_{n=-\infty}^{\infty} \frac{1}{2}\left(1 + \cos(\tfrac{\pi}{2}n)\right) = \infty$$

$$P = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{4}\sum_{n=0}^{3} \cos^2(\tfrac{\pi}{4}n) = \frac{1}{4}\left(1 + \frac{1}{2} + 0 + \frac{1}{2}\right) = \frac{2}{4} = \frac{1}{2}$$

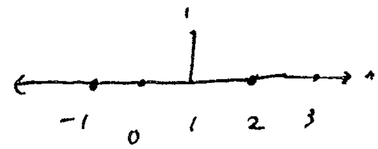
6) a: $\sum_{n=-\infty}^{\infty} n^2 \delta[n-3] = n^2\Big|_{n=3} = 9$

b: $x[n] = n^2 \delta[n-T] \qquad T = 0, 1, \ldots, 4$

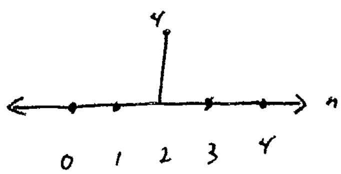
$T = 0: \quad x[n] = 0$



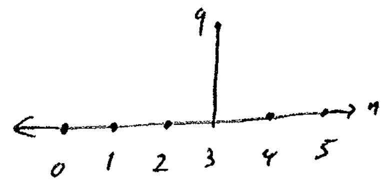
$T = 1: \quad x[n] = \delta[n-1]$



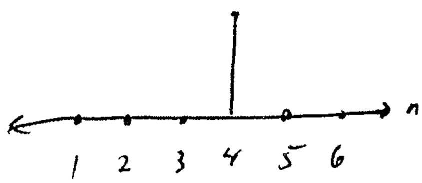
$T = 2: \quad x[n] = 4\delta[n-2]$



$T = 3 \quad x[n] = 9\delta[n-3]$



$T = 4: \quad x[n] = 16\,\delta[n-4]$



c: $\sum_{k=-\infty}^{\infty} u[k]\,\delta[n-k] = u[k]\Big|_{k=n} \qquad \forall n \in \mathbb{Z}$

$\Rightarrow \sum_{k=-\infty}^{\infty} u[k]\,\delta[n-k] = u[n]$

d: $\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] = x[k]\Big|_{k=n} \qquad \forall n \in \mathbb{Z}$

$\Rightarrow \sum_{k=-\infty}^{\infty} \delta[n-k]\,x[k] = x[n]$