ECE 301 HW 11 Solutions

1) a: $x[n] = a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n = \begin{cases} \frac{1}{1-az^{-1}} & |az^{-1}| < 1 \\ \infty & \text{ow} \end{cases}$$

$$X(z) = \frac{z}{z-a} \qquad ROC = \{ |z| > |a| \}$$

b: $x[n] = a^n u[-n]$

$$X(z) = \sum_{n=-\infty}^{0} a^n z^{-n} \qquad \text{Let } k = -n$$

$$= \sum_{k=0}^{\infty} (a^{-1} z)^k = \begin{cases} \frac{1}{1-\frac{1}{a}z} & |\frac{z}{a}| < 1 \\ \infty & \text{ow} \end{cases}$$

$$X(z) = \frac{1}{1-\frac{1}{a}z} \qquad ROC = \{ |z| < |a| \}$$

c: $x[n] = a^{|n|}$ $\qquad a < 1$

$$X(z) = \sum_{n=-\infty}^{\infty} a^{|n|} z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=-\infty}^{0} a^{-n} z^{-n} - 1$$

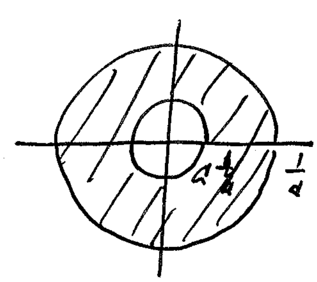
Let $k = -n$

$$= \sum_{n=0}^{\infty} (az^{-1})^n + \sum_{k=0}^{\infty} (az)^k - 1$$

$$= \frac{1}{1-az^{-1}} + \frac{1}{1-az} - 1$$

$$ROC = \{ |az^{-1}| < 1 \} \cap \{ |az| < 1 \} = \{ |z| > |a| \} \cap \{ |z| < |\tfrac{1}{a}| \}$$

$$= \frac{z}{z-a} + \frac{-\frac{1}{a}}{z-\frac{1}{a}} - 1$$



since $a < 1$

$$= \frac{z^2 - \frac{1}{a}z - \frac{1}{a}z + 1 - z^2 + \frac{1}{a}z + az - 1}{(z-a)(z-\frac{1}{a})}$$

$$= \frac{z(a - \frac{1}{a})}{(z-a)(z-\frac{1}{a})}$$

2) $X(z) = \frac{1}{1+a z^{-1}}$

a: right-sided sequence $\Rightarrow$ ROC $= \{ |z| > |a| \}$

$x[n] = (-a)^n u[n]$

b: left-sided sequence $\Rightarrow$ ROC $= \{ |z| < \frac{1}{|a|} \}$

$x[n] = (\frac{1}{a})^n u[-n]$

c: For $|a| < 1$, the right-sided inverse is stable

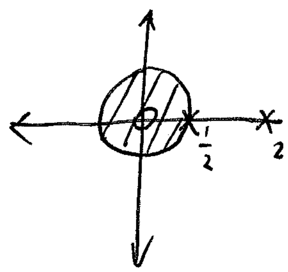
For $|a| > 1$, the left-sided inverse is stable

3) $H(z) = \dfrac{1}{(2z^{-1}-1)(\frac{1}{2}z^{-1}-1)} = \dfrac{z^2}{(z-2)(z-\frac{1}{2})}$

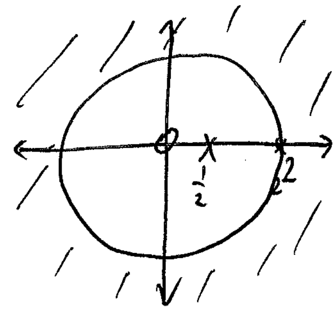
a: 2 poles @ $z = \frac{1}{2}, 2$

2 zeroes @ $z = 0$

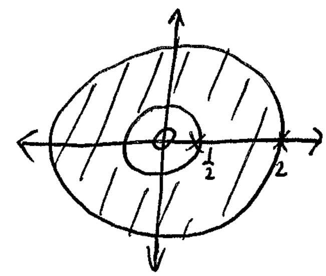
b: $ROC_1 = \{ |z| < \frac{1}{2} \}$



$ROC_2 = \{ |z| > 2 \}$



$ROC_3 = \{ \frac{1}{2} < |z| < 2 \}$



c: left-sided

d: unstable

right-sided (causal)
unstable

two-sided
stable

e: $H(z) = \dfrac{A}{(2z^{-1}-1)} + \dfrac{B}{(\frac{1}{2}z^{-1}-1)}$

$A(\frac{1}{2}z^{-1}-1) + B(2z^{-1}-1) = 1$

$z^{-1}(\frac{1}{2}A + 2B) = 0\,z^{-1}$ $\qquad B = -\frac{1}{4}A$ $\qquad B = \frac{1}{3}$

$-A - B = 1$ $\qquad -A + \frac{1}{4}A = 1$ $\qquad A = -\frac{4}{3}$

$H(z) = \frac{1}{3}\dfrac{1}{\frac{1}{2}z^{-1}-1} - \frac{4}{3}\dfrac{1}{2z^{-1}-1}$

Case 1: $\{|z| < \frac{1}{2}\} = ROC$

$$\{|z| < \tfrac{1}{2}\} = \{|z| < 2\} \cap \{|z| < \tfrac{1}{2}\}$$

$$\alpha^n u[-n-1] \Longleftrightarrow \frac{-1}{1-\alpha z^{-1}} \quad ROC = \{|z| < |\alpha|\}$$

$$\left(\tfrac{1}{2}\right)^n u[-n-1] \Longleftrightarrow \frac{-1}{1-\frac{1}{2}z^{-1}} \quad ROC = \{|z| < \tfrac{1}{2}\}$$

$$(2)^n u[-n-1] \Longleftrightarrow \frac{-1}{1-2z^{-1}} \quad ROC = \{|z| < 2\}$$

$$H(z) = \frac{1}{3}\frac{-1}{1-\frac{1}{2}z^{-1}} - \frac{4}{3}\frac{-1}{1-2z^{-1}} \qquad ROC = \{|z| < \tfrac{1}{2}\}$$

$$h[n] = \tfrac{1}{3}\left(\tfrac{1}{2}\right)^n u[-n-1] - \tfrac{4}{3}(2)^n u[-n-1]$$

Case 2: $ROC = \{|z| > 2\}$

$$\{|z| > 2\} = \{|z| > \tfrac{1}{2}\} \cap \{|z| > 2\}$$

$$\alpha^n u[n] \Longleftrightarrow \frac{1}{1-\alpha z^{-1}} \quad ROC = \{|z| > |\alpha|\}$$

$$\left(\tfrac{1}{2}\right)^n u[n] \Longleftrightarrow \frac{1}{1-\frac{1}{2}z^{-1}} \quad ROC = \{|z| > \tfrac{1}{2}\}$$

$$(2)^n u[n] \Longleftrightarrow \frac{1}{1-2z^{-1}} \quad ROC = \{|z| > 2\}$$

$$H(z) = -\frac{1}{3}\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{4}{3}\frac{1}{1-2z^{-1}} \qquad ROC = \{|z| > 2\}$$

$$h[n] = -\tfrac{1}{3}\left(\tfrac{1}{2}\right)^n u[n] + \tfrac{4}{3}(2)^n u[n]$$

Case 3: $ROC = \{ \frac{1}{2} < |z| < 2 \}$

$$\left(\frac{1}{2}\right)^n u[n] \Leftrightarrow \frac{1}{1-\frac{1}{2}z^{-1}} \quad ROC = \{ |z| > \frac{1}{2} \}$$

$$(2)^n u[-n-1] \Leftrightarrow \frac{-1}{1-2z^{-1}} \quad ROC = \{ |z| < 2 \}$$

$$\{ \tfrac{1}{2} < |z| < 2 \} = \{ |z| > \tfrac{1}{2} \} \cap \{ |z| < 2 \}$$

$$H(z) = -\frac{1}{3} \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{4}{3} \frac{-1}{1-2z^{-1}}$$

$$h[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1]$$

4) $y[n] = a y[n-1] + x[n]$

a: $Y(z) = a z^{-1} Y(z) + X(z)$

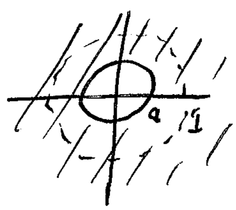
$Y(z)(1 - a z^{-1}) = X(z)$

$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}$ zero @ $z = 0$

pole @ $z = a$

b: $ROC = \{ |z| > a \}$

$h[n] = a^n u[n]$ stability $\Longleftrightarrow$ $|a| < 1$



c: $ROC = \{ |z| < a \}$

$h[n] = \left(\frac{1}{a}\right)^n u[-n]$ stability $\Longleftrightarrow$ $|a| > 1$