EE301 Homework #5: Orthonormal Transformations

Problem 1 *Orthogonality and normalized functions.*
The *inner product* of two functions over the interval $t \in (a, b)$ is defined as

$$\langle f(t), g(t) \rangle = \int_a^b f(t) g^*(t)\, dt$$

where $g^*(t)$ denotes the complex conjugate of $g(t)$. The functions $f(t)$ and $g(t)$ are called *orthogonal* if $\langle f(t), g(t) \rangle = 0$.

A function $f(t)$ is said to be *normalized* if

$$\langle f(t), f(t) \rangle = \int_a^b |f(t)|^2\, dt = 1$$

(a) Are the functions $\sin(m\omega_0 t)$ and $\sin(n\omega_0 t)$ orthogonal over the interval $t \in (0, T)$ where $T = 2\pi/\omega_0$ and $m \neq n$?

(b) Let $x(t)$ be an arbitrary real-valued signal, and let

$$x_o(t) = \frac{x(t) - x(-t)}{2}$$

and

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

be the the odd and even parts of $x(t)$, respectively. Show that $x_o(t)$ and $x_e(t)$ are orthogonal over the interval $t \in (-T, T)$ for any T.

(c) Find the value of a so that the functions $f(t) = t$ and $g(t) = t - at^2$ are orthogonal over the interval $t \in (0, 1)$. Determine the real-valued constants c_1 and c_2 so that $\frac{f(t)}{c_1}$ and $\frac{g(t)}{c_2}$ are both normalized.

Problem 2 *Orthonormal set of functions.*
A set of functions $\{\phi_k(t)\}$, where k is any integer, is called an *orthonormal set* if (i) $\phi_k(t)$ and $\phi_m(t)$ are orthogonal for $k \neq m$ and (ii) all functions in $\{\phi_k(t)\}$ are normalized.

For each of the following problems, check if the given set of functions form an orthonormal set over the specified interval. If the functions are not normalized determine the constant C so that the set $\{\phi_k(t)/C\}$ is orthonormal.

(a) $\phi_k(t) = \frac{e^{jk\omega_0 t}}{\sqrt{T}}$ for $\omega_0 = \frac{2\pi}{T}$ and $t \in (0, T)$.

(b) $\phi_k(t) = \cos(k\omega_0 t)$ for $\omega_0 = \frac{2\pi}{T}$ and $t \in (0, T)$.

Problem 3 *Function expansion using orthonormal functions.*
Given a complete orthonormal basis $\{\phi_k(t)\}_{k=-\infty}^{\infty}$ over the interval $t \in (a, b)$, then we can express a function $x(t)$ on the interval (a, b) as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t) \tag{1}$$

Show that the coefficients, a_k, in the above expression can be determined using the formula

$$a_m = \int_a^b x(t)\phi_m^*(t)\,dt$$

(Hint: Multiply both sides of Equation 1 with $\phi_m^*(t)$ and integrate both sides.)

Problem 4 *Fourier series expansion.*
As you have shown in Problem 2, the set $\frac{e^{jk\omega_0 t}}{\sqrt{T}}$ is orthonormal over the interval $t \in (0, T)$. Using the result of Problem 3 we see that we can expand a given function $x(t)$, which is periodic with period T using this set as

$$x(t) = \frac{1}{\sqrt{T}} \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

This representation is called the *orthonormal Fourier series representation* of $x(t)$. [1]
Consider the periodic function with period $T = 2$

$$x(t) = \sum_{k=-\infty}^{\infty} (t - kT)[u(t - kT) - u(t - kT - 2)]$$

(e.g. for $0 \le t \le 2$, $x(t) = t$). Compute the values of the coefficients, a_k, for all integers k.

Problem 5 *Parseval's formula.*
For a signal expressed using Equation 1 show that

$$\int_a^b |x(t)|^2\,dt = \sum_{m=-\infty}^{\infty} |a_m|^2$$

This important result is known as the Parseval's formula. Note that the left side is the energy in $x(t)$.

[1] In class, we defined the conventional Fourier series expansion using the basis functions $e^{jk\omega_0 t}$ which are orthogonal, but not normal. In this case, the Fourier series coefficients are given by $\frac{a_k}{\sqrt{T}}$.