EE301 Homework #4: LTI System Properties

Problem 1 *Properties of convolution.*

(a) Consider a CT LTI system $y(t) = x(t) * h(t)$. Show the input $\frac{dx(t)}{d(t)}$ results in the output $\frac{dy(t)}{d(t)}$.

(b) Consider the DT LTI system $y_n = x_n * h_n$. Prove that

$$\sum_{n=-\infty}^{\infty} y_n = \left(\sum_{n=-\infty}^{\infty} x_n\right)\left(\sum_{n=-\infty}^{\infty} h_n\right)$$

(c) Consider a CT LTI system $y(t) = x(t) * h(t)$. Prove that if $x(t)$ is periodic with period T, then $y(t)$ is also periodic with period T.

Problem 2 *Properties of convolution.*
Let x_n be a signal which is nonzero only in the interval $0 \le n < M$ and h_n be a signal which is nonzero only in the interval $0 \le n < N$.

(a) Determine the interval $L_1 \le n \le L_2$ over which $y_n = x_n * h_n$ is nonzero. Express L_1 and L_2 in terms of M and N.

(b) Verify the result in the the previous part by analytically computing the convolution of the signals $x_n = u_n - u_{n-5}$ and $h_n = 2(u_n - u_{n-3})$.

(c) Verify the result in the the previous part by graphically computing the convolution of the signals $x_n = u_n - u_{n-5}$ and $h_n = 2(u_n - u_{n-2})$.

Problem 3 - *Properties of LTI systems.*
Prove the following properties.

(a) The commutative property of DT convolution, that is, $x_n * y_n = y_n * x_n$

(b) The associative property of DT convolution, that is, $(x_n * y_n) * z_n = x_n * (y_n * z_n)$

(c) The distributive property of DT convolution, that is, $x_n * (y_n + z_n) = x_n * y_n + x_n * z_n$

(d) Let h_n be the impulse response of a DT system. Then the system is causal if and only if $h_n = 0$ for $n < 0$.

Problem 4 *Causal and Stable LTI systems.*
For the following discrete-time and continuous-time LTI systems, determine whether each system is causal and/or stable. Justify your answers.

(a) $h_n = (\frac{1}{2})^n u_{-n}$

(b) $h_n = (-\frac{1}{2})^n u_n + (1.01)^n u_{n-1}$

(c) $h(t) = e^{2t}u(-1-t)$

(d) $h(t) = te^{-t}u(t)$

Problem 5 - *LTI differential equations.*
Determine the impulse response for the following system under the assumption that the system is initially at rest.

$$\frac{dy(t)}{dt} = -ay(t) + x(t)$$

Problem 6 *DT difference equations*
Consider the DT LTI system described by the equation

$$y[n] = \frac{1}{2}y[n-1] + x[n]$$

where $\lim_{n\to-\infty} y[n] = 0$.

(a) Compute the impulse response of the system.

(b) Express the system in the form $y[n] = x[n] * h[n]$.

(c) Find the output when the input is given by $x[n] = u[n]$.

(d) Find the output when the input is given by $x[n] = 1$.

Problem 7 *System response to a complex exponential input.*
For the following continuous-time and discrete-time systems with the given input and output, determine whether the system is definitely *not* LTI.

(a) $S_1[e^{j7t}] = te^{j7t}$

(b) $S_2[e^{j7t}] = e^{j7(t-2)}$

(c) $S_3[e^{j7t}] = \sin(7t)$

(d) $S_4[e^{j\pi n/4}] = e^{j\pi n/4}u_n$

(e) $S_5[e^{j\pi n/4}] = e^{j3\pi n/4}$

(f) $S_6[e^{j\pi n/4}] = 2e^{3\pi/4}e^{j\pi n/4}$

Problem 8 *System response to a complex exponential input.*
Let $y(n) = S\left[x(n)\right]$ be a LTI system with discrete-time input $x(n)$, discrete-time output $y(n)$, and impulse response $h(n)$.

(a) Write an explicit expression for the output in terms of the input and the impulse response.

(b) If the input to the systems is $x(n) = e^{j\omega n}$, then show that the output must have the form

$$y(n) = C(\omega)e^{j\omega n}$$

where $C(\omega)$ is a complex value that is a function of ω. Also, calculate an explicit expression for $C(\omega)$ in terms of the inputs response.

(c) Show that if $h(t)$ is real valued, then for all $\omega \in \Re$

$$C(-\omega) = C^*(\omega)$$

or equivalently that if $C(\omega) = A(\omega)e^{j\theta}$, then

$$\begin{aligned} A(\omega) &= A(\omega) \\ \theta(\omega) &= -\theta(\omega) \ . \end{aligned}$$

(d) Use the result of part (c) above to compute the output $y(n)$ when $x(n) = \cos(\omega n)$.

(e) Use the result of part (c) above to compute the output $y(n)$ when $x(n) = B\cos(\omega n + \phi)$.

(f) Use the result of part (c) above to compute the output $y(n)$ when $x(n) = \sin(\omega n)$.

(g) Use the result of part (c) above to compute the output $y(n)$ when $x(n) = B\sin(\omega n + \phi)$.