EE 301 Final Exam
December 14, Fall 2009

Name: Key

Instructions:

- Follow all instructions carefully!
- This is a 120 minute exam containing **5 problems** totaling 100 points.
- You **may not** use a calculator.
- You **may not** use any notes, books or other references.
- You may only keep pencils, pens and erasers at your desk during the exam.

Good Luck.

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- Continuous Time Fourier Series (CTFS)

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\frac{2\pi}{T}t} dt \\ x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} \end{aligned}$$

- CTFS Properties

$$x(t-t_o) \overset{CTFS}{\Leftrightarrow} a_k e^{-jk\frac{2\pi}{T}t_o}$$

For $x(t)$ real valued $a_k = a^*_{-k}$

$$\frac{dx(t)}{dt} \overset{CTFS}{\Leftrightarrow} jk\frac{2\pi}{T}a_k$$

$$x(t) = \int_{-T/2}^{T/2} x(\tau)y(t-\tau)d\tau \overset{CTFS}{\Leftrightarrow} Ta_k b_k$$

$$x(t)y(t) \overset{CTFS}{\Leftrightarrow} \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

- CTFT

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-\omega)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(\omega)e^{-j\omega t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|}X(\omega/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} 2\pi x(-\omega)$$

$$x(t)e^{j\omega_0 t} \overset{CTFT}{\Leftrightarrow} X(\omega-\omega_0)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{2\pi}X(\omega) * Y(\omega)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(\omega)Y(\omega)$$

$$\frac{dx(t)}{dt} \overset{CTFT}{\Leftrightarrow} j\omega X(\omega)$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

If $x(t) \overset{CTFS}{\Leftrightarrow} a_k$ then

$$x(t) \overset{CTFT}{\Leftrightarrow} \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - 2\pi k/T)$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(\omega/(2\pi))$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(\omega/(2\pi))$$

For $a > 0$

$$\frac{1}{(n-1)!}t^{n-1}e^{-at}u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j\omega + a)^n}$$

$$\sum_{k-\infty}^{\infty} \delta(t-kT) \overset{CTFT}{\Leftrightarrow} \frac{1}{T}\sum_{k-\infty}^{\infty} 2\pi\delta(\omega-2\pi k/T)$$

- DFT

$$\begin{aligned} X_k &= \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \\ x(n) &= \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N} \end{aligned}$$

- DTFT

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - ae^{-j\omega}}$$

- Z-transform

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x(n) z^{-n} \\
x(n) &= \frac{1}{2\pi j} \oint_{C \in ROC} X(z) z^{n-1} dz
\end{aligned}
$$

- Sampling and Reconstruction

$$
\begin{aligned}
y(n) &= x(nT) \\
Y(\omega) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{T}\right) \\
s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t - kT) \\
S(\omega) &= Y(\omega T)
\end{aligned}
$$

Name: ______________________________

Problem 1.(20pt) *Z-transforms and LTI systems*

Let $h(n) = a^n u(n)$ be the impulse response of an LTI system.

a) Calculate the Z-transform, $H(z)$, and its associated Region of Convergence (ROC).

b) Specify the pole(s) and zero(s) of $H(z)$, and sketch the pole(s), zero(s), and ROC on the complex plane.

c) If 1 is contained in the ROC, then what do you know about the LTI system? For what values of a is it guaranteed that 1 is contained in the ROC?

d) Determine a difference equation which has an impulse response of $h(n)$, and draw it's flow diagram.

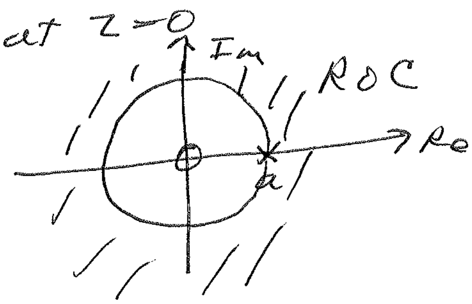
a) $$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} = \sum_{n=0}^{\infty} a^n z^n = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1-az^{-1}}$$

b) $$H(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

1 pole at $z=a$

1 zero at $z=0$



c) If $1 \in ROC \Rightarrow \sum_{n=-\infty}^{\infty} |h(n)| < \infty \Rightarrow$ ~~BIBO~~ stable

$(1 \in ROC) \Leftrightarrow |a| < 1$

d) $$y(n) = a\, y(n-1) + x(n)$$

$$(1-az^{-1})Y(z) = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-az^{-1}}$$

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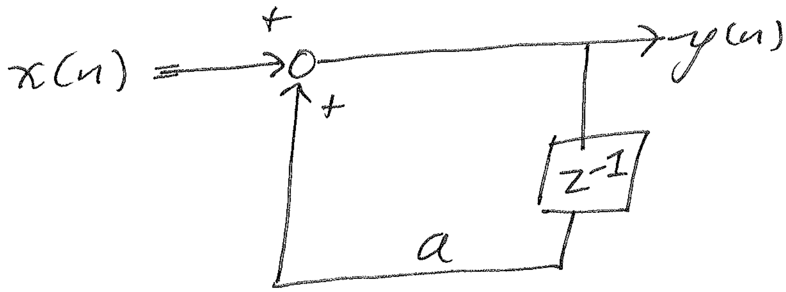

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Problem 2.(20pt) *Sampling and reconstruction*

Consider a sampling system with input $x(t)$ and sampled signal $y(n) = x(nT)$. The sampled signal is then put into a digital-to-analog converter (DAC) to produce an output of $s(t)$, and the resulting output is filtered with a perfect low pass filter to form a reconstructed signal $f(t)$. Assume that the low pass filter has a cutoff frequency of $\omega_s/2$, where $\omega_s = \frac{2\pi}{T}$, and a pass-band gain of 1.

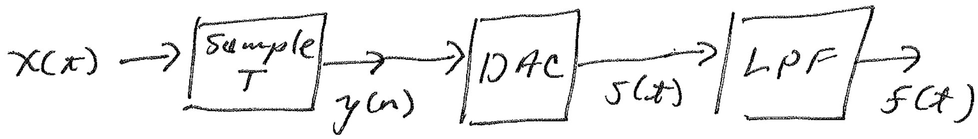
Furthermore, assume that $x(t)$ is band-limited to frequency $\omega_c < \omega_s$.

a) Give an expression for $Y(\omega)$, the DTFT of $y(n)$, in terms of $X(\omega)$, the CTFT of $x(t)$. Sketch typical signals for $Y(\omega)$ and $X(\omega)$.

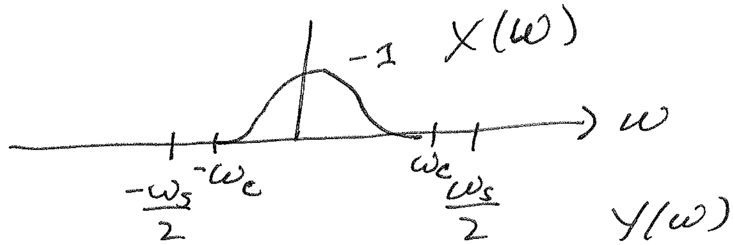
b) Calculate an expression for $S(\omega)$, the CTFT of $s(t)$, in terms of $X(\omega)$, the CTFT of $x(t)$.

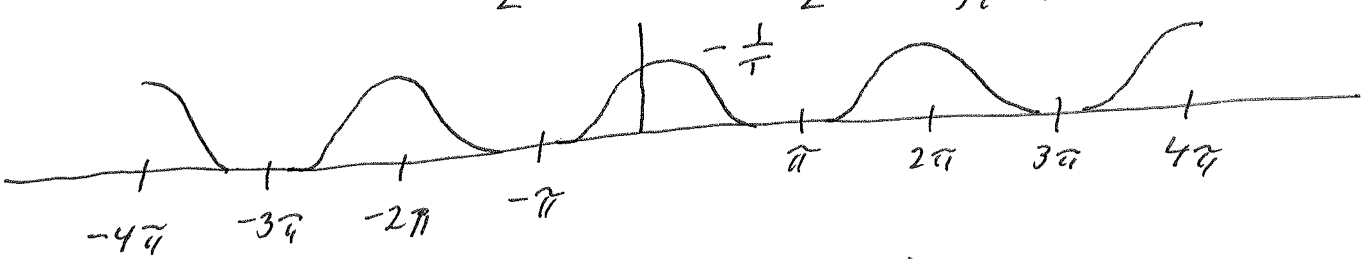
c) Sketch typical signals for $S(\omega)$ and $F(\omega)$.

d) Derive an expression for $F(\omega)$, the CTFT of $f(t)$, in terms of $X(\omega)$, the CTFT of $x(t)$.



a) $$Y(\omega) = \frac{1}{T}\sum_{K=-\infty}^{\infty} X\left(\frac{\omega - 2\pi K}{T}\right)$$





b) $S(\omega) = P(\omega)\, Y(T\omega)$

where $p(t) = \text{rect}(t/T)$

ignores time shift of DAC.

$P(\omega) = T\,\text{sinc}(T\omega/2\pi)$

$= T\,\text{sinc}(\omega/\omega_s)$

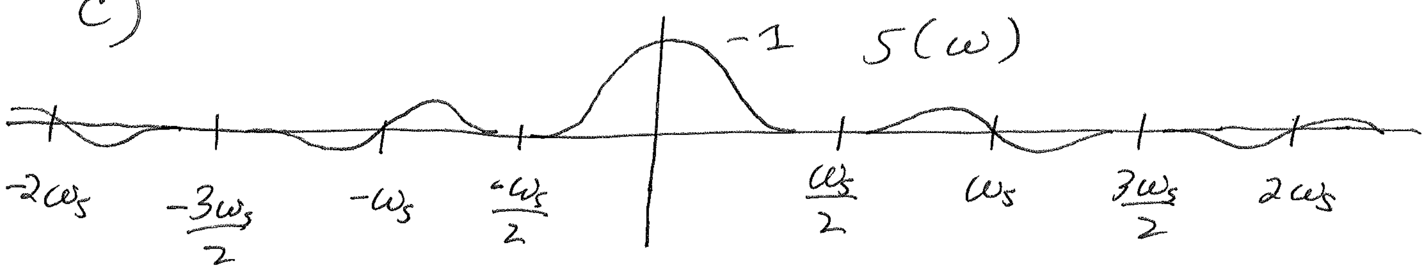
Name: ____________________

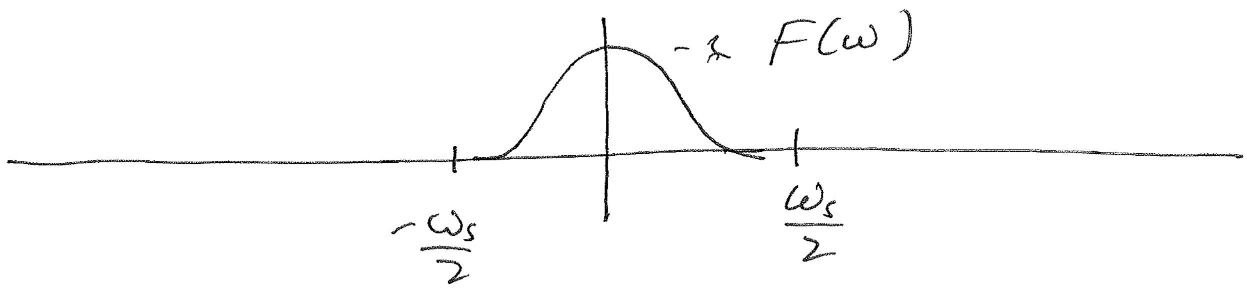
$$S(\omega) = P(\omega)\,Y(T\omega)$$

$$= T\,\mathrm{sinc}(\omega/\omega_s)\,\frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{T\omega - 2\pi k}{T}\right)$$

$$= \mathrm{sinc}\left(\frac{\omega}{\omega_s}\right)\sum_{k=-\infty}^{\infty} X(\omega - \omega_s k)$$

c)





d) $F(\omega) = H(\omega)\,S(\omega)$

perfect low pass filter

$$= \mathrm{sinc}\left(\frac{\omega}{\omega_s}\right) X(\omega)$$

Name: ______________________________

Problem 3.(20pt) *Orthonormal Basis Functions*

Let $\phi_k(t) = e^{j2\pi kt}$ for k an integer be functions on the interval $[\frac{1}{2}, \frac{1}{2}]$ with associated inner product given by

$$\langle \phi_m, \phi_l \rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi_m(t)\phi_l^*(t)dt \ .$$

a) Prove that the functions $\{\phi_k\}_{k=-\infty}^{\infty}$ are normal.
b) Prove that the functions $\{\phi_k\}_{k=-\infty}^{\infty}$ are orthogonal.
c) Assume that $x(t)$ is a real valued signal that has the form

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, \phi_k(t) \ .$$

Derive an expression for the coefficients a_k in terms of the signal $x(t)$.
d) Show that Parseval's theorem applies, i.e. that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

a) $$< \phi_K , \phi_K > = \int_{-1/2}^{1/2} e^{j2\pi K t} e^{-j2\pi K t} dt$$

$$= \int_{-1/2}^{1/2} 1 \, dt = 1$$

b) $$< \phi_m , \phi_l > = \int_{-1/2}^{1/2} e^{j2\pi m t} e^{-j2\pi l t} dt$$

$$= \int_{-1/2}^{1/2} e^{j2\pi (m-l)t} dt = \frac{1}{\pi(m-l)} \frac{1}{2j} e^{j2\pi(m-l)} \Big|_{-1/2}^{1/2}$$

$$= \frac{1}{\pi(m-l)} \frac{e^{j\pi(m-l)} - e^{-j\pi(m-l)}}{2j}$$

$$= \frac{1}{\pi(m-l)} \sin(\pi(m-l)) = \operatorname{sinc}(\cancel{\pi}(m-l))$$

$$= \delta(m-l)$$

Name: ________________________________

c) $\langle x, \phi_m \rangle = \left\langle \sum_{k=\infty}^{\infty} a_k \phi_k , \phi_m \right\rangle$

$$= \sum_{k=\infty}^{\infty} a_k \langle \phi_k , \phi_m \rangle = \sum_{k=-\infty}^{\infty} a_k \; \delta(k-m)$$

$$= a_m$$

$$a_k = \langle x, \phi_k \rangle$$

$$= \int_{-1/2}^{1/2} x(t)\, e^{-j2\pi k t}\, dt$$

d) $\displaystyle\int_{-1/2}^{1/2} |x(t)|^2 dt = \int_{-1/2}^{1/2} x(t)\; x^*(t)\, dt$

$$= \int_{-1/2}^{1/2} \left(\sum_k a_k \phi_k(t)\right)\left(\sum_\ell a_\ell \phi_\ell(t)\right)^* dt$$

$$= \sum_k \sum_\ell a_k a_\ell^* \underbrace{\int_{-1/2}^{1/2} \phi_k(t)\, \phi_\ell^*(t)\, dt}_{\delta(k-\ell)}$$

$$= \sum_k a_k a_k^* = \sum_k |a_k|^2$$

Name: ______________________________

Problem 4.(20pt) *Properties LTI systems*

Consider the discrete-time LTI system $y(n) = S[x(n)]$ with input $x(n)$ and output $y(n)$.

a) Prove that

$$y(n) = \sum_{k=-\infty}^{\infty} h(n-k)\, x(k)$$

where $h(n)$ is the impulse response of the system.

b) Prove that convolution is commutative, i.e. that

$$\sum_{k=-\infty}^{\infty} h(n-k)\, x(k) = \sum_{k=-\infty}^{\infty} x(n-k)\, h(k)$$

c) Use the results of parts a) and b) above to prove that for any input with the form

$$x(n) = e^{j\omega n}$$

then the output has the form $y(n) = H(\omega)e^{j\omega n}$, and derive an explicit expression for the function $H(\omega)$.

d) Show that if the impulse response is real valued, then $H(-\omega)$ is the complex conjugate of $H(\omega)$.

a) $$x(n) = \sum_{K} x(K)\,\delta(n-K)$$

$$y(n) = S[x(n)] = S\left[\sum_{K} x(K)\,\delta(n-K)\right]$$

$$= \sum_{K} x(K)\, S[\delta(n-K)]$$

$$= \sum_{K} x(K)\, h(n-K)$$

b) $$\sum_{K} h(n-K)\, x(K) = \sum_{m=-\infty}^{\infty} h(m)\, x(n-m)$$

$$m \triangleq n-K$$

$$K = n-m$$

Name: ________________________________

c) $$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, e^{j\omega(n-k)}$$

$$= e^{j\omega n} \sum_{k=-\infty}^{\infty} h(k)\, e^{-j\omega k}$$

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n}$$

d) $$H(-\omega) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j(-\omega)n}$$

$$= \sum_{n=-\infty}^{\infty} h(n)\, e^{j\omega n}$$

$$= \left(\sum_{n=-\infty}^{\infty} h(n)\, e^{j\omega n}\right)^{*}$$

$$= (H(\omega))^{*}$$

Name: ______________________________

Problem 5.(20pt) *Analysis of LTI system*

Consider the continuous-time LTI system $y(t) = S[x(t)]$ with input $x(t)$ and output $y(t)$. Let

$$x(t) = 1 + \cos(\pi t) + \cos(2\pi t)$$

and let the impulse response be given by

$$h(t) = (\text{sinc}(t))^2 \; .$$

a) Determine the fundamental period of $x(t)$.
b) Determine the Continuous Time Fourier Series (CTFS) coefficients, a_k, for the signal $x(t)$.
c) Calculate and sketch a plot of the frequency response of the system, $H(\omega)$.
d) Calculate the output signal $y(t)$.

a) $T = 1/2$

b) $x(t) = 1 + \frac{1}{2} e^{j\pi t} + \frac{1}{2} e^{-j\pi t} + \frac{1}{2} e^{j2\pi t} + \frac{1}{2} e^{-j2\pi t}$

$$a_k = \begin{cases} 1 & \text{for } k=0 \\ \frac{1}{2} & \text{for } k=\pm 1 \\ \frac{1}{2} & \text{for } k = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

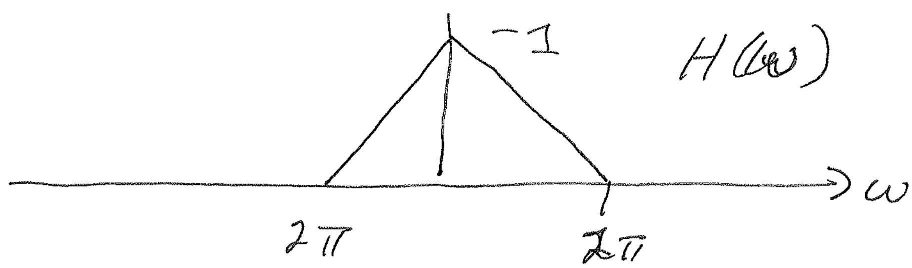
Name: ______________________________

c) $H(\omega) = CTFT\{h(t)\}$

$= \frac{1}{2\pi} CTFT\{sinc(t)\} * CTFT\{sinc(t)\}$

$= \frac{1}{2\pi} rect(\omega/2\pi) * rect(\omega/2\pi)$

$= \Lambda(\omega/2\pi)$



d) $S[1] = 1\,H(0) = 1$

$S[e^{j\pi t}] = e^{j\pi t} H(\pi) = \frac{1}{2} e^{j\pi t}$

$S[e^{j2\pi t}] = e^{j2\pi t} H(2\pi) = 0$

$\cancel{S[e^{j4\pi t}]}$

$y(t) = 1 + \frac{1}{2}\cos(\pi t)$