EE 301 Midterm Exam #3
November 20, Fall 2009

Name: Key

Instructions:

- Follow all instructions carefully!
- This is a 50 minute exam containing **four** problems totaling 100 points.
- You **may not** use a calculator.
- You **may not** use any notes, books or other references.
- You may only keep pencils, pens and erasers at your desk during the exam.

Good Luck.

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- Continuous Time Fourier Series (CTFS)

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\frac{2\pi}{T}t} dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

- CTFS Properties

$$x(t-t_o) \overset{CTFS}{\Leftrightarrow} a_k e^{-jk\frac{2\pi}{T}t_o}$$

For $x(t)$ real valued $a_k = a^*_{-k}$

$$\frac{dx(t)}{dt} \overset{CTFS}{\Leftrightarrow} jk\frac{2\pi}{T}a_k$$

$$x(t) = \int_{-T/2}^{T/2} x(\tau)y(t-\tau)d\tau \overset{CTFS}{\Leftrightarrow} Ta_k b_k$$

$$x(t)y(t) \overset{CTFS}{\Leftrightarrow} \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

- CTFT

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-\omega)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(\omega)e^{-j\omega t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|}X(\omega/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} 2\pi x(-\omega)$$

$$x(t)e^{j\omega_0 t} \overset{CTFT}{\Leftrightarrow} X(\omega-\omega_0)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{2\pi}X(\omega) * Y(\omega)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(\omega)Y(\omega)$$

$$\frac{dx(t)}{dt} \overset{CTFT}{\Leftrightarrow} j\omega X(\omega)$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

If $x(t) \overset{CTFS}{\Leftrightarrow} a_k$ then

$$x(t) \overset{CTFT}{\Leftrightarrow} \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - 2\pi k/T)$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(\omega/(2\pi))$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(\omega/(2\pi))$$

For $a > 0$

$$\frac{1}{(n-1)!}t^{n-1}e^{-at}u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j\omega + a)^n}$$

$$\sum_{k-\infty}^{\infty} \delta(t-kT) \overset{CTFT}{\Leftrightarrow} \frac{1}{T}\sum_{k-\infty}^{\infty} 2\pi\delta(\omega-2\pi k/T)$$

- DFT

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

$$x(n) = \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

- DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - ae^{-j\omega}}$$

- Sampling and Reconstruction

$$y(n) = x(nT)$$

$$Y(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{T}\right)$$

$$s(t) = \sum_{k=-\infty}^{\infty} y(k)\delta(t-kT)$$

$$S(\omega) = Y(\omega T)$$

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Problem 1.(25pt) *Calculating the DFT*

Calculate $X(k)$ the N point DFT for each of the following signals. Assume that N is even, $0 \le m < N$, and and calculate a solution that is correct for $1 \le k < N$.

a) $x(n) = e^{j2\pi mn/N}$ for $0 \le n < N$

b) $x(n) = e^{-j2\pi mn/N}$ for $0 \le n < N$

c) $x(n) = \cos\left(\frac{2\pi mn}{N} + \theta\right)$ for $0 \le n < N$

d) $x(n) = (-1)^n$ for $0 \le n < N$

a) $X(K) = \delta(K-m)$

b) $X(K) = \delta(K-(N-m))$

c) $x(n) = \dfrac{e^{j(\frac{2\pi m}{N}n+\theta)} + e^{-j(\frac{2\pi m}{N}n+\theta)}}{2}$

$= \dfrac{e^{j\theta}}{2} e^{j(\frac{2\pi m}{N}n)} + \dfrac{e^{-j\theta}}{2} e^{-j(\frac{2\pi m}{N}n)}$

$X(K) = \dfrac{e^{j\theta}}{2}\delta(K-m) + \dfrac{e^{-j\theta}}{2}\delta(K-(N-m))$

d) $x(n) = (-1)^n = e^{j2\pi\frac{N/2}{N}n}$

$X(K) = \delta(K - N/2)$

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Problem 2.(25pt) *Properties of the DFT*

Consider the N point DFT representation of the DT signal $x(n)$ given by

$$x(n) = \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} .$$

a) Prove that $x(n)$ is periodic with period N.

b) Prove that the coefficients $X(k)$ can be calculated as

$$X(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} .$$

c) Prove that

$$\sum_{n=0}^{N-1} |x(n)|^2 = N \sum_{k=0}^{N-1} |X(k)|^2$$

a) $$x(n+N) = \sum_{k=0}^{N-1} X(k) e^{j\left(\frac{2\pi k}{N}\right)(n+N)}$$

$$= \sum_{k=0}^{N-1} X(k) e^{j\left(\frac{2\pi k}{N} n\right)} \underbrace{e^{j2\pi k}}_{=1}$$

$$= x(n)$$

b) $$\frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} \sum_{l=0}^{N-1} X(l) e^{j\left(\frac{2\pi l n}{N}\right)} \cdot e^{-j\left(\frac{2\pi kn}{N}\right)}$$

$$= \frac{1}{N}\sum_{l=0}^{N-1} X(l) \underbrace{\sum_{n=0}^{N-1} e^{j\frac{2\pi(l-k)}{N} n}}_{A}$$

$$A = \frac{1 - e^{j\frac{2\pi(l-k)}{N} N}}{1 - e^{j\frac{2\pi(l-k)}{N}}} = N\delta(l-k) \quad \text{for } \begin{matrix} 0 \le l < N \\ 0 \le k < N \end{matrix}$$

$$= \frac{1}{N}\sum_{l=0}^{N-1} X(l) N \delta(l-k) = X(k)$$

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c) $\sum_{n=0}^{N-1} |x(n)|^2 = \sum_{n=0}^{N-1} x(n)\, x^*(n)$

$= \sum_{n=0}^{N-1} \left(\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi k n}{N}} \right) \left(\sum_{l=0}^{N-1} X^*(l) e^{-j\frac{2\pi l n}{N}} \right)$

$= \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} X(k) X^*(l) \underbrace{\sum_{n=0}^{N-1} e^{j\frac{2\pi (k-l) n}{N}}}_{\text{same as in point a)}}$

$= N\,\delta(k-l)$ for $0 \le k < N$, $0 \le l < N$

$= \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} X(k) X^*(l)\, N\, \delta(k-l)$

$= N \sum_{k=0}^{N-1} X(k) X^*(k) = N \sum_{k=0}^{N-1} |X(k)|^2$

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Problem 3.(25pt) *Discrete-time System Analysis*

Consider the discrete-time LTI system which obeys the difference equation

$$y(n) = by(n-1) + x(n)$$

where $|b| < 1$ with input $x(n) = a^n u(n)$ and $a \neq b$.

a) Calculate the frequency response of the system $H(\omega)$.

b) Calculate the impulse response of the system $h(n)$.

c) Calculate the DTFT of the output, $Y(\omega)$.

d) Calculate the output, $y(n)$.

a) $Y(\omega) = b e^{-j\omega} Y(\omega) + X(\omega)$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1 - b e^{-j\omega}}$$

b) $h(n) = DTFT^{-1}\{H(\omega)\}$

$$= b^n u(n)$$

c) $X(\omega) = \dfrac{1}{1 - a e^{-j\omega}}$

$$Y(\omega) = H(\omega)X(\omega) = \frac{1}{(1 - a e^{-j\omega})(1 - b e^{-j\omega})}$$

d)

$$Y(\omega) = \frac{A}{1 - a e^{-j\omega}} + \frac{B}{1 - b e^{-j\omega}}$$

let $z = e^{j\omega}$

$$\frac{A}{1 - a z^{-1}} + \frac{B}{1 - b z^{-1}} = \frac{1}{(1 - a z^{-1})(1 - b z^{-1})}$$

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$$A = \frac{1}{1-ba^{-1}} = \frac{a}{a-b}$$

$$B = \frac{1}{1-ab^{-1}} = \frac{b}{b-a}$$

$$Y(\omega) = \frac{a}{a-b}\left(\frac{1}{1-ae^{-j\omega}}\right) - \frac{b}{a-b}\left(\frac{1}{1-be^{-j\omega}}\right)$$

$$h(n) = \frac{a}{a-b}\,a^n u(n) - \frac{b}{a-b}\,b^n u(n)$$

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Problem 4.(25pt) *LTI Systems*

Consider the system with input $x(t)$ and output $s(t)$ specified by

$$s(t) = x(t)\left(\sum_{k=-\infty}^{\infty} \delta(t-kT)\right),$$

and assume that $x(t) = \text{sinc}(t)$.

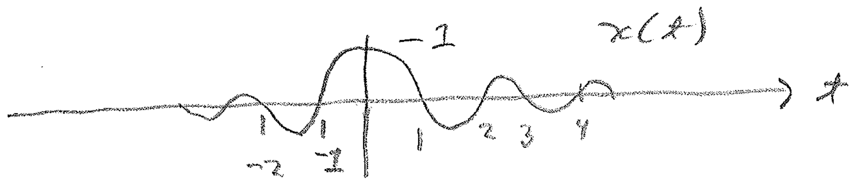
a) Give an expression for $X(\omega)$, the CTFT of $x(t)$; and sketch both $x(t)$ and $X(\omega)$.

b) Sketch $s(t)$ for $T = 1/2$, $T = 1$, and $T = 3/2$.

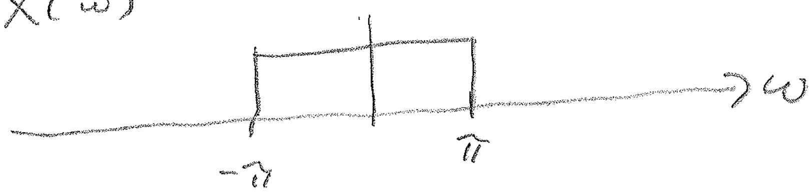
c) Calculate $S(\omega)$, the CTFT of $s(t)$.

d) Sketch $S(\omega)$ for $T = 1/2$, $T = 1$, and $T = 3/2$.

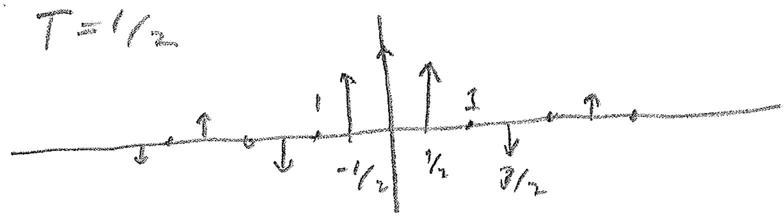
a) $x(t) = \text{sinc}(t)$ $\quad X(\omega) = \text{rect}\left(\frac{\omega}{2\pi}\right)$



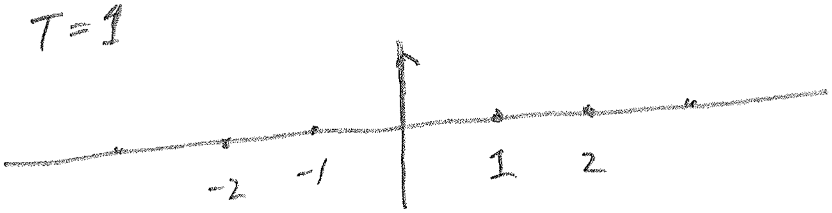
$X(\omega)$



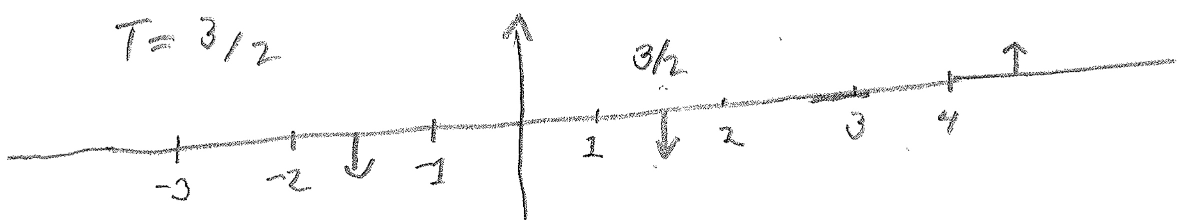
b) $T = 1/2$



$T = 1$



$T = 3/2$



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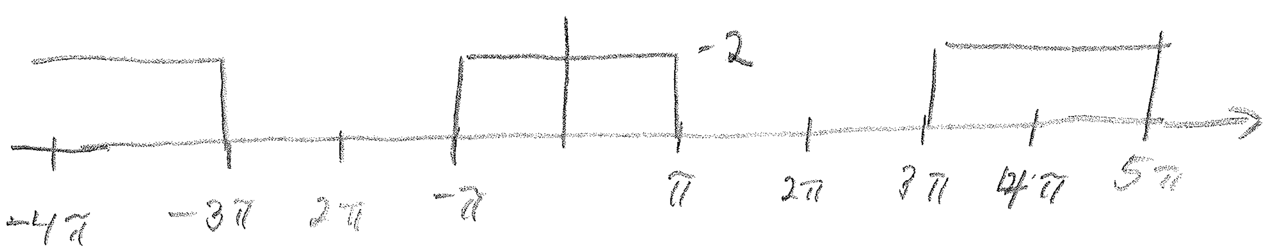
c) $$\underbrace{\sum_{k=-\infty}^{\infty} \delta(t-kT)}_{y(t)} \overset{CTFT}{\Longleftrightarrow} \underbrace{\frac{1}{T} \sum_{k=-\infty}^{\infty} 2\pi\, \delta\left(\omega - \frac{2\pi k}{T}\right)}_{Y(\omega)}$$

$$S(\omega) = \frac{1}{2\pi} X(\omega) * Y(\omega)$$

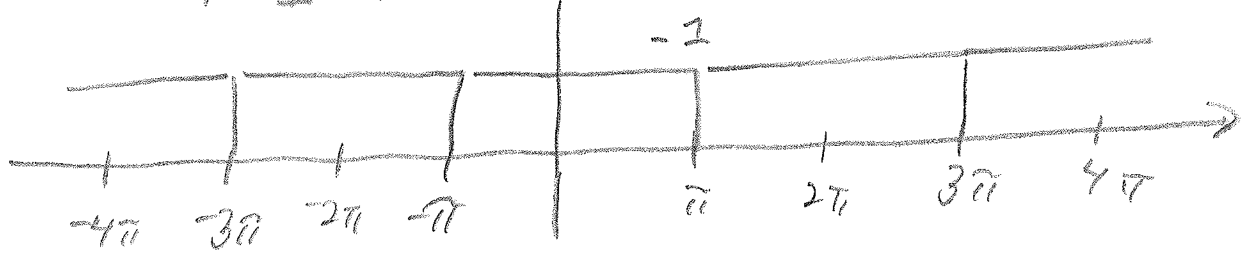
$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X\left(\omega - \frac{2\pi k}{T}\right)$$

d) $T = 1/2 \quad \frac{1}{T} = 2$

$S(\omega)$



$T = 1 \quad \frac{1}{T} = 1$



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$T = 3/2 \quad \frac{1}{T} = \frac{2}{3}$