EE 301 Midterm Exam #2

October 23, Fall 2009

Name: Key

Instructions:

- Follow all instructions carefully!
- This is a 50 minute exam containing **four** problems totaling 100 points.
- You **may not** use a calculator.
- You **may not** use any notes, books or other references.
- You may only keep pencils, pens and erasers at your desk during the exam.

Good Luck.

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1 - |t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- Continuous Time Fourier Series (CTFS)

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\frac{2\pi}{T}t} dt \\ x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} \end{aligned}$$

- CTFS Properties

$$x(t-t_o) \overset{CTFS}{\Leftrightarrow} a_k e^{-jk\frac{2\pi}{T}t_o}$$

For $x(t)$ real valued $a_k = a_{-k}^*$

$$\frac{dx(t)}{dt} \overset{CTFS}{\Leftrightarrow} jk\frac{2\pi}{T} a_k$$

$$x(t) = \int_{-T/2}^{T/2} x(\tau) y(t-\tau) d\tau \overset{CTFS}{\Leftrightarrow} T a_k b_k$$

$$x(t)y(t) \overset{CTFS}{\Leftrightarrow} \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

- CTFT

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\ x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-\omega)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(\omega) e^{-j\omega t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|} X(\omega/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} 2\pi x(-\omega)$$

$$x(t) e^{j\omega_0 t} \overset{CTFT}{\Leftrightarrow} X(\omega - \omega_0)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{2\pi} X(\omega) * Y(\omega)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(\omega) Y(\omega)$$

$$\frac{dx(t)}{dt} \overset{CTFT}{\Leftrightarrow} j\omega X(\omega)$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

If $x(t) \overset{CTFS}{\Leftrightarrow} a_k$ then

$$x(t) \overset{CTFT}{\Leftrightarrow} \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - 2\pi k/T)$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(\omega/(2\pi))$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(\omega/(2\pi))$$

For $a > 0$

$$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j\omega + a)^n}$$

$$\sum_{k-\infty}^{\infty} \delta(t - kT) \overset{CTFT}{\Leftrightarrow} \frac{1}{T} \sum_{k-\infty}^{\infty} 2\pi \delta(\omega - 2\pi k/T)$$

- DFT

$$\begin{aligned} X_k &= \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \\ x(n) &= \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N} \end{aligned}$$

- DTFT

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1 - a e^{-j\omega}}$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(\omega) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega - 2\pi k}{T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k) \delta(t - kT) \\ S(\omega) &= Y(\omega T) \end{aligned}$$

Name: ______________________________

Problem 1. (25pt) *Sine wave inputs to LTI systems*

Let $y(t) = S[x(t)]$ be an LTI system with input $x(t)$, output $y(t)$, and real valued impulse response $h(t)$. Determine the output, $y(t)$, in the following cases.

a) $x(t) = e^{j\omega_o t}$.

b) $x(t) = e^{-j\omega_o t}$.

c) $x(t) = \cos(\omega_o t)$.

d) $x(t) = B\cos(\omega_o t + \phi)$, where B and ϕ are real values.

a) $$y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau) e^{j\omega_o (t-\tau)}\, d\tau$$

$$= e^{j\omega_o t} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-j\omega_o \tau}\, d\tau}_{C}$$

$$y(t) = C e^{j\omega_o t}$$

b) $$y(t) = C^* e^{-j\omega_o t}$$

$$C = \int_{-\infty}^{\infty} h(\tau) e^{-j(-\omega_o)\tau}\, d\tau = \left[\int_{-\infty}^{\infty} h(\tau) e^{-j\omega_o \tau}\, d\tau\right]^* = C^*$$

c) $$x(t) = \frac{1}{2}\left[e^{j\omega_o t} + e^{-j\omega_o t}\right]$$

$$y(t) = \frac{1}{2}\left[C e^{j\omega_o t} + C^* e^{-j\omega_o t}\right] \qquad \text{Let } C \triangleq A e^{j\theta}$$

$$= \frac{1}{2} A \left[e^{j(\omega_o t + \theta)} + e^{-j(\omega_o t - \theta)}\right]$$

$$= A\cos(\omega_o t + \theta)$$

Name: ________________________________

d) By Linearity

$$S[B\cos(\omega_0 t)] = B\,S[\cos(\omega_0 t)]$$

$$= BA\cos(\omega_0 t + \theta)$$

By time invariance

$$S[B\cos(\omega_0 t + \phi)] = S\left[B\cos\left(\omega_0\left(t + \frac{\phi}{\omega_0}\right)\right)\right]$$

$$= AB\cos\left(\omega_0\left(t + \frac{\phi}{\omega_0}\right) + \theta\right)$$

$$= AB\cos(\omega_0 t + \theta + \phi)$$

Name: ____________________

Problem 2. (25pt) *The continuous time Fourier series*

For each example, $x(t)$ is a period signal with period T and associated frequency $\omega_o = \frac{2\pi}{T}$.

a) Calculate the Fourier series coefficients, a_k, for the signal, $x(t) = e^{j2\omega_o t} + e^{j5\omega_o t}$.

b) Calculate the Fourier series coefficients, a_k, for the signal, $x(t) = \cos(2\omega_o t)$.

c) Calculate the Fourier series coefficients, a_k, for the signal, $x(t) = B\cos(2\omega_o t + \phi)$.

d) Calculate a closed form expression for the signal, $x(t)$, when $x(t) = \sum_{k=-\infty}^{\infty} e^{jk\omega_o t}$.

a) $$a_K = \begin{cases} 1 & \text{if } K = 2 \\ 1 & \text{if } K = 5 \\ 0 & \text{otherwise} \end{cases} \qquad a_K = \delta(K-2) + \delta(K-5)$$

b) $$a_K = \begin{cases} \frac{1}{2} & \text{if } K = 2 \\ \frac{1}{2} & \text{if } K = -2 \\ 0 & \text{otherwise} \end{cases} \qquad a_K = \frac{1}{2}\left(\delta(K-2) + \delta(K+2)\right)$$

c) $$x(t) = \frac{B}{2}\left(e^{j(2\omega_o t + \phi)} + e^{-j(2\omega_o t + \phi)}\right)$$

$$= \frac{B}{2} e^{j\phi} e^{j2\omega_o t} + \frac{B}{2} e^{-j\phi} e^{-j2\omega_o t}$$

$$a_K = \begin{cases} \frac{B}{2} e^{j\phi} & \text{if } K = 2 \\ \frac{B}{2} e^{-j\phi} & \text{if } K = -2 \\ 0 & \text{otherwise} \end{cases}$$

$$a_K = \frac{B}{2} e^{j\phi}\delta(K-2) + \frac{B}{2} e^{-j\phi}\delta(K+2)$$

Name: ______________________

d) We know that $x(t) = \delta(t) \quad |t| < \frac{T}{2}$

$$\overset{CTFS}{\Longleftrightarrow} \quad a_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} \, dt = \frac{1}{T}$$

So

$$x(t) = T\delta(t) \overset{CTFS}{\Longleftrightarrow} a_k = 1$$

$$x(t) = T\delta(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$T\delta(t) = \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$$

Name: ______________________________

Problem 3.(25pt) *Orthonormal Basis Functions*

Let $\phi_k(t) = e^{j2\pi kt}$ for k an integer be functions on the interval $[0,1]$ with associated inner product given by

$$\langle \phi_k, \phi_l \rangle = \int_0^1 \phi_k(t)\phi_l^*(t)dt \ .$$

a) Prove that the functions $\{\phi_k\}_{k=-\infty}^{\infty}$ are normal.
b) Prove that the functions $\{\phi_k\}_{k=-\infty}^{\infty}$ are orthogonal.
c) Assume that $x(t)$ is a real valued signal that has the form

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, \phi_k(t) \ .$$

Then show that Parseval's theorem applies, i.e. that

$$\int_0^1 |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

d) Let $\tilde{x}(t)$ be formed by filtering the signal $x(t)$ so that only the first M harmonics are retained. Then the RMS error between these two signals is defined as

$$\text{RMS} = \sqrt{\int_0^1 |x(t) - \tilde{x}(t)|^2 \, dt}$$

Then calculate an expression for the RMS error between $x(t)$ and $\tilde{x}(t)$ in terms of the coefficients a_k.

$\omega_0 = 2\pi$

a) $\langle \phi_K, \phi_K \rangle = \int_0^1 e^{jK\omega_0 t} e^{-jK\omega_0 t} dt$

$= \int_0^1 1 \, dt = \underline{1}$

b) $\langle \phi_K, \phi_\ell \rangle = \int_0^1 e^{jK\omega_0 t} e^{-j\ell\omega_0 t} dt$

$= \int_0^1 e^{j\omega_0 (K-\ell) t} dt$

$= \frac{1}{j\omega_0 (K-\ell)} e^{j\underbrace{\omega_0}_{2\pi}(K-\ell)t} \Big|_0^1 = \frac{1}{j2\pi(K-\ell)}(1-1)$

$= 0$

Name: ______________________________

c) $\int_0^1 |x(t)|^2 dt = \langle x, x \rangle$

$= \langle \sum_{k=-\infty}^{\infty} a_k \phi_k , \sum_{\ell=-\infty}^{\infty} a_\ell \phi_\ell \rangle$

$= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_k a_\ell^* \langle \phi_k, \phi_\ell \rangle$

$= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} a_k a_\ell^* \delta_{k-\ell} = \sum_{k=-\infty}^{\infty} |a_k|^2$

d) $\int_0^1 |x(t) - \tilde{x}(t)|^2 dt = \sum_{k=-\infty}^{-M-1} |a_k|^2 + \sum_{k=M+1}^{\infty} |a_k|^2$

$= 2 \sum_{k=M+1}^{\infty} |a_k|^2$

$RMSE = \sqrt{2 \sum_{k=M+1}^{\infty} |a_k|^2}$

Name: ______________________________

Problem 4.(25pt) *Periodic inputs to LTI systems*

Let $x(t)$ be a period signal with period T and associated frequency $\omega_o = \frac{2\pi}{T}$; and let a_k be the associated Fourier series expansion coefficients for $x(t)$.

The signal $x(t)$ is used as the input to an LTI system with output $y(t)$ and real valued impulse response h(t).

a) Show that $y(t)$ is periodic with period T.

b) Derive an expression for, b_k, the Fourier series coefficients of $y(t)$ in terms of the a_k's and coefficients computed from the impulse response, $h(t)$.

c) Calculate an expression for b_k when $h(t) = \text{rect}\,(t/T)$.

d) Use the result of c) above to compute an explicit expression for $y(t)$.

a) $$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

$$y(t+T) = \int_{-\infty}^{\infty} x(t+T-\tau)\, h(\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

$$= y(t)$$

b) $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

$$y(t) = S[x(t)] = S\left[\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}\right]$$

$$= \sum_{k=-\infty}^{\infty} a_k S\left[e^{jk\omega_o t}\right] = \sum_{k=-\infty}^{\infty} a_k c_k e^{jk\omega_o t}$$

where $c_k = \int_{-\infty}^{\infty} h(t) e^{-jk\omega_o t}\, dt$

So

$$b_k = a_k c_k$$

Name: ____________________

c) $$c_n = \int_{-\infty}^{\infty} rect(t/T)\, e^{-jk\omega_0 t}\, dt$$

$$= \int_{-T/2}^{T/2} e^{-jk\omega_0 t}\, dt = \frac{1}{-jk\omega_0} e^{-jk\omega_0 t}\Big|_{-T/2}^{T/2}$$

$$= \frac{1}{-jk\omega_0}\left[e^{-j\pi k} - e^{j\pi k}\right]$$

$$= T\frac{1}{\pi k}\,\frac{e^{j\pi k} - e^{-j\pi k}}{2j}$$

$$= T\, sinc(k) = T\,\delta(k)$$

$$b_k = a_k\,\delta(k) = \begin{cases} a_0 & \text{for } k=0 \\ 0 & \text{otherwise} \end{cases}$$

d) $$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t} = b_0\, 1 = a_0$$

$$y(t) = a_0$$