EE 301 Midterm Exam #1

September 25, Fall 2009

Name: Key

Instructions:

- Follow all instructions carefully!
- This is a 50 minute exam containing **four** problems totaling 100 points.
- You **may not** use a calculator.
- You **may not** use any notes, books or other references.
- You may only keep pencils, pens and erasers at your desk during the exam.

Good Luck.

Fact Sheet

- Function definitions

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- Continuous Time Fourier Series (CTFS)

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\frac{2\pi}{T}t} dt \\ x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{-jk\frac{2\pi}{T}t} \end{aligned}$$

- CTFS Properties

$$x(t-t_o) \overset{CTFS}{\Leftrightarrow} a_k e^{-jk\frac{2\pi}{T}t_o}$$

For $x(t)$ real valued $a_k = a_{-k}^*$

$$\frac{dx(t)}{dt} \overset{CTFS}{\Leftrightarrow} jk\frac{2\pi}{T}a_k$$

$$x(t) = \int_{-T/2}^{T/2} x(\tau)y(t-\tau)d\tau \overset{CTFS}{\Leftrightarrow} Ta_k b_k$$

$$x(t)y(t) \overset{CTFS}{\Leftrightarrow} \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

- CTFT

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \\ x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega \end{aligned}$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-\omega)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(\omega)e^{-j\omega t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|}X(\omega/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} 2\pi x(-\omega)$$

$$x(t)e^{j\omega_0 t} \overset{CTFT}{\Leftrightarrow} X(\omega-\omega_0)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{2\pi}X(\omega) * Y(\omega)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(\omega)Y(\omega)$$

$$\frac{dx(t)}{dt} \overset{CTFT}{\Leftrightarrow} j\omega X(\omega)$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

If $x(t) \overset{CTFS}{\Leftrightarrow} a_k$ then

$$x(t) \overset{CTFT}{\Leftrightarrow} \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - 2\pi k/T)$$

- CTFT pairs

$$\mathrm{sinc}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{rect}(\omega/(2\pi))$$

$$\mathrm{rect}(t) \overset{CTFT}{\Leftrightarrow} \mathrm{sinc}(\omega/(2\pi))$$

For $a > 0$

$$\frac{1}{(n-1)!}t^{n-1}e^{-at}u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j\omega+a)^n}$$

$$\sum_{k-\infty}^{\infty} \delta(t-kT) \overset{CTFT}{\Leftrightarrow} \frac{1}{T}\sum_{k-\infty}^{\infty} 2\pi\delta(\omega-2\pi k/T)$$

- DFT

$$\begin{aligned} X_k &= \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \\ x(n) &= \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N} \end{aligned}$$

- DTFT

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n} d\omega \end{aligned}$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1-ae^{-j\omega}}$$

- Sampling and Reconstruction

$$\begin{aligned} y(n) &= x(nT) \\ Y(\omega) &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega-2\pi k}{T}\right) \\ s(t) &= \sum_{k=-\infty}^{\infty} y(k)\delta(t-kT) \\ S(\omega) &= Y(\omega T) \end{aligned}$$

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Problem 1.(25pt) *Formal Logic*

Consider the function $x(t)$ for $t \in \Re$.

We say that the function $x(t)$ is bounded if there exists an $M \in \Re$ such that for all $t \in \Re$, $|x(t)| \leq M$.

Alternatively, we say that a function $x(t)$ is unbounded if it is not bounded.

a) Give a formal logical expression for the definition of bounded.

b) Give a formal logical expression for the definition of unbounded.

c) Prove that the function $x(t) = e^t$ is either bounded or unbounded.

a) $\exists M > 0$ s.t. $\forall t \in \mathbb{R}$ $|x(t)| \leq M$

b) $\forall M > 0$ $\exists t \in \mathbb{R}$ s.t. $|x(t)| > M$

c) $x(t) = e^t$ is unbounded

proof.

Choose any $M > 0$ then choose

$t = \log M + 1$

Then it is the case that

$$|x(t)| = |e^t| = |e^{\log M + 1}|$$

$$= |Me|$$

$$= Me > M$$

So we have that for any $M > 0$, if we choose $t = \log M + 1$, then

$$|x(t)| > M$$

$\Rightarrow$ unbounded.

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Problem 2.(25pt) *System Properties*

For each of the following continuous-time systems with input $x(t)$ and output $y(t)$, prove or disprove that it is i) linear; ii) time-invariant; iii) BIBO stable.

a) $y(t) = e^t x(t)$

b) $y(t) = e^{x(t)}$

c) $y(t) = \int_0^t x(\tau) d\tau$

a) <u>Linear</u>

$$S[\alpha x_1(t) + \beta x_2(t)] = e^t(\alpha x_1(t) + \beta x_2(t))$$

$$= \alpha e^t x_1(t) + \beta e^t x_2(t) = \alpha S[x_1(t)] + \beta S[x_2(t)]$$

<u>Time varying</u>

$$S[\delta(t)] = e^t \delta(t) = \delta(t)$$

$$S[\delta(t-1)] = e^t \delta(t-1) = e^1 \delta(t-1) \neq \delta(t-1)$$

<u>Not BIBO stable</u>

Choose $x(t) = 1$. Then $y(t) = e^t$.

Since $x(t)$ is bounded, and $y(t)$ is not bounded $\Rightarrow$ not BIBO

b) <u>Non linear</u> Let $x(t) = 1$

$$S[x(t)] = e^{x(t)} = e^1 = e$$

$$S[2x(t)] = e^2 \neq 2e = 2S[x(t)]$$

<u>TI</u> $\forall x(t)$

$$S[x(t)] = e^{x(t)} \triangleq y(t)$$

$$S[x(t-T)] = e^{x(t-T)} = y(t-T)$$

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BIBO

Let $x(t)$ be bounded, then $\exists\, M>0$ s.t.

$$|x(t)|<M \Rightarrow |y(t)| = |e^{x(t)}| < e^{M}$$

$$\Rightarrow y(t) \text{ is bounded.}$$

c) Linear

$$S[\alpha x_1(t) + \beta x_2(t)] = \int_0^t [\alpha x_1(\tau) + \beta x_2(\tau)]\, d\tau$$

$$= \alpha \int_0^t x_1(\tau)\, d\tau + \beta \int_0^t x_2(\tau)\, d\tau$$

$$= \alpha S[x_1(t)] + \beta S[x_2(t)]$$

Time-varying

Let $x(t) = \text{rect}(t)$

$y(t) = S[x(t)]$ $\quad z(t) = S[x(t-1/2)]$

Question $z(t) \stackrel{?}{=} y(t-1/2)$

Notice that $z(1/2) = \int_0^{1/2} \text{rect}(\tau - 1/2)\, d\tau$

$$= 1/2$$

But $y(1/2 - 1/2) = y(0) = \int_0^0 \text{rect}(\tau)\, d\tau = 0$

$$1/2 \neq 0$$

Not BIBO

Let $x(t) = 1$. Then $y(t) = \int_0^t 1\, d\tau = t$

But $y(t) = t$ is not bounded.

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Problem 3.(25pt) *Impulse Response*

Let $y(n) = S[x(n)]$ be an LTI discrete-time system with input $x(n)$ and output $y(n)$.

a) Prove that

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

(Hint: The easiest approach is to just apply the definition of $\delta(n)$.)

b) Prove that

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

where $h(n) \triangleq S[\delta(n)]$. Justify each step of your proof.

a) For $k \neq n$ $\delta(n-k) = 0$, so the only nonzero term in the sum is $x(n)\,\delta(n-n) = x(n)$.

So $\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) = x(n)\delta(n-n) = x(n)$

b) $S[x(n)] = S\left[\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)\right]$

$\overset{\text{Linear}}{=} \sum_{k=-\infty}^{\infty} x(k)\, S[\delta(n-k)]$

$\overset{\text{Time Invariant}}{=} \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$

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Problem 4.(25pt) *Difference Equations*

Consider the discrete time system $y(n) = S[x(n)]$, which is governed by the difference equation

$$y(n) = ay(n-1) + x(n)$$

where $x(n)$ is the input and $y(n)$ is the output.

It can be proven that this is a LTI system. So for the remainder of the problem, you may assume this is true.

a) Verify that if $x(n) = \delta(n)$ then $y(n) = a^n u(n)$.

b) For what values of a is the system BIBO stable?

c) Express the output, $y(n)$, as an explicit function of the input, $x(n)$.

d) Calculate the output, $y(n)$, when $x(n) = u(n)$.

a) $a^n u(n) = a(a^{n-1}u(n-1)) + x(n)$

$= a^n u(n-1) + x(n)$

$a^n (u(n) - u(n-1)) = x(n)$

$a^n \delta(n) = x(n)$

So $x(n) = a^n \delta(n) = \delta(n)$

$a^0 = 1$

b) $h(n) = a^n u(n) \Leftarrow$ impulse response

BIBO $\Leftrightarrow \sum_n |h(n)| < \infty$

$= \sum_{n=0}^{\infty} |a^n| = \sum_{n=0}^{\infty} |a|^n < \infty$

$\Leftrightarrow |a| < 1$

BIBO $\Leftrightarrow |a| < 1$

c) $y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k) = \sum_{k=-\infty}^{\infty} x(k) a^{n-k} u(n-k)$

$= \sum_{k=-\infty}^{n} x(k) a^{n-k}$

$y(n) = \sum_{k=0}^{\infty} a^k x(n-k)$

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d) $y(n) = \sum_{k=0}^{\infty} a^k u(n-k)$

$$= \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} u(n)$$