EE 301 Final Exam

December 14, Fall 2009

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Instructions:

- Follow all instructions carefully!
- This is a 120 minute exam containing **5 problems** totaling 100 points.
- You **may not** use a calculator.
- You **may not** use any notes, books or other references.
- You may only keep pencils, pens and erasers at your desk during the exam.

Good Luck.

Fact Sheet

- Function definitions

$$\text{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- Continuous Time Fourier Series (CTFS)

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\frac{2\pi}{T}t} dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

- CTFS Properties

$$x(t-t_o) \overset{CTFS}{\Leftrightarrow} a_k e^{-jk\frac{2\pi}{T}t_o}$$

For $x(t)$ real valued $a_k = a_{-k}^*$

$$\frac{dx(t)}{dt} \overset{CTFS}{\Leftrightarrow} jk\frac{2\pi}{T}a_k$$

$$x(t) = \int_{-T/2}^{T/2} x(\tau)y(t-\tau)d\tau \overset{CTFS}{\Leftrightarrow} Ta_k b_k$$

$$x(t)y(t) \overset{CTFS}{\Leftrightarrow} \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

- CTFT

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-\omega)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(\omega)e^{-j\omega t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|}X(\omega/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} 2\pi x(-\omega)$$

$$x(t)e^{j\omega_0 t} \overset{CTFT}{\Leftrightarrow} X(\omega-\omega_0)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{2\pi}X(\omega) * Y(\omega)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(\omega)Y(\omega)$$

$$\frac{dx(t)}{dt} \overset{CTFT}{\Leftrightarrow} j\omega X(\omega)$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

If $x(t) \overset{CTFS}{\Leftrightarrow} a_k$ then

$$x(t) \overset{CTFT}{\Leftrightarrow} \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - 2\pi k/T)$$

- CTFT pairs

$$\text{sinc}(t) \overset{CTFT}{\Leftrightarrow} \text{rect}(\omega/(2\pi))$$

$$\text{rect}(t) \overset{CTFT}{\Leftrightarrow} \text{sinc}(\omega/(2\pi))$$

For $a > 0$

$$\frac{1}{(n-1)!}t^{n-1}e^{-at}u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j\omega+a)^n}$$

$$\sum_{k-\infty}^{\infty} \delta(t-kT) \overset{CTFT}{\Leftrightarrow} \frac{1}{T}\sum_{k-\infty}^{\infty} 2\pi\delta(\omega-2\pi k/T)$$

- DFT

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

$$x(n) = \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

- DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n} d\omega$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1-ae^{-j\omega}}$$

- Sampling and Reconstruction

$$y(n) = x(nT)$$

$$Y(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega-2\pi k}{T}\right)$$

$$s(t) = \sum_{k=-\infty}^{\infty} y(k)\delta(t-kT)$$

$$S(\omega) = Y(\omega T)$$

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Problem 1.(20pt) *Z-transforms and LTI systems*

Let $h(n) = a^n u(n)$ be the impulse response of an LTI system.

a) Calculate the Z-transform, $H(z)$, and its associated Region of Convergence (ROC).

b) Specify the pole(s) and zero(s) of $H(z)$, and sketch the pole(s), zero(s), and ROC on the complex plane.

c) If 1 is contained in the ROC, then what do you know about the LTI system? For what values of a is it guaranteed that 1 is contained in the ROC?

d) Determine a difference equation which has an impulse response of $h(n)$, and draw it's flow diagram.

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Problem 2.(20pt) *Sampling and reconstruction*

Consider a sampling system with input $x(t)$ and sampled signal $y(n) = x(nT)$. The sampled signal is then put into a digital-to-analog converter (DAC) to produce an output of $s(t)$, and the resulting output is filtered with a perfect low pass filter to form a reconstructed signal $f(t)$.

You many make the following assumptions:

- Assume that the low pass filter has a cutoff frequency of $\omega_s/2$, where $\omega_s = \frac{2\pi}{T}$, and a pass-band gain of 1.
- Assume that $x(t)$ is band-limited to frequency $\omega_c < \frac{\omega_s}{2}$.
- You may ignore the one half sample delay of the DAC.

a) Give an expression for $Y(\omega)$, the DTFT of $y(n)$, in terms of $X(\omega)$, the CTFT of $x(t)$. Sketch typical signals for $Y(\omega)$ and $X(\omega)$.

b) Calculate an expression for $S(\omega)$, the CTFT of $s(t)$, in terms of $X(\omega)$, the CTFT of $x(t)$.

c) Sketch typical signals for $S(\omega)$ and $F(\omega)$.

d) Derive an expression for $F(\omega)$, the CTFT of $f(t)$, in terms of $X(\omega)$, the CTFT of $x(t)$.

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Problem 3.(20pt) *Orthonormal Basis Functions*

Let $\phi_k(t) = e^{j2\pi kt}$ for k an integer be functions on the interval $[\frac{1}{2}, \frac{1}{2}]$ with associated inner product given by

$$\langle \phi_m, \phi_l \rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi_m(t)\phi_l^*(t)dt \ .$$

a) Prove that the functions $\{\phi_k\}_{k=-\infty}^{\infty}$ are normal.
b) Prove that the functions $\{\phi_k\}_{k=-\infty}^{\infty}$ are orthogonal.
c) Assume that $x(t)$ is a real valued signal that has the form

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, \phi_k(t) \ .$$

Derive an expression for the coefficients a_k in terms of the signal $x(t)$.
d) Show that Parseval's theorem applies, i.e. that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |x(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

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Problem 4.(20pt) *Properties LTI systems*

Consider the discrete-time LTI system $y(n) = S[x(n)]$ with input $x(n)$ and output $y(n)$.

a) Prove that

$$y(n) = \sum_{k=-\infty}^{\infty} h(n-k)\, x(k)$$

where $h(n)$ is the impulse response of the system.

b) Prove that convolution is commutative, i.e. that

$$\sum_{k=-\infty}^{\infty} h(n-k)\, x(k) = \sum_{k=-\infty}^{\infty} x(n-k)\, h(k)$$

c) Use the results of parts a) and b) above to prove that for any input with the form

$$x(n) = e^{j\omega n}$$

then the output has the form $y(n) = H(\omega)e^{j\omega n}$, and derive an explicit expression for the function $H(\omega)$.

d) Show that if the impulse response is real valued, then $H(-\omega)$ is the complex conjugate of $H(\omega)$.

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Problem 5.(20pt) *Analysis of LTI system*

Consider the continuous-time LTI system $y(t) = S[x(t)]$ with input $x(t)$ and output $y(t)$. Let

$$x(t) = 1 + \cos(\pi t) + \cos(2\pi t)$$

and let the impulse response be given by

$$h(t) = (\text{sinc}(t))^2 \ .$$

a) Determine the fundamental period of $x(t)$.

b) Determine the Continuous Time Fourier Series (CTFS) coefficients, a_k, for the signal $x(t)$.

c) Calculate and sketch a plot of the frequency response of the system, $H(\omega)$.

d) Calculate the output signal $y(t)$.

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