EE 301 Midterm Exam #1

September 25, Fall 2009

Name: ___________________________________

Instructions:

- Follow all instructions carefully!
- This is a 50 minute exam containing **four** problems totaling 100 points.
- You **may not** use a calculator.
- You **may not** use any notes, books or other references.
- You may only keep pencils, pens and erasers at your desk during the exam.

Good Luck.

Fact Sheet

- Function definitions

$$\text{rect}(t) \triangleq \begin{cases} 1 & \text{for } |t| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$\Lambda(t) \triangleq \begin{cases} 1-|t| & \text{for } |t| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

- Continuous Time Fourier Series (CTFS)

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\frac{2\pi}{T}t}dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

- CTFS Properties

$$x(t-t_o) \overset{CTFS}{\Leftrightarrow} a_k e^{-jk\frac{2\pi}{T}t_o}$$

For $x(t)$ real valued $a_k = a^*_{-k}$

$$\frac{dx(t)}{dt} \overset{CTFS}{\Leftrightarrow} jk\frac{2\pi}{T}a_k$$

$$x(t) = \int_{-T/2}^{T/2} x(\tau)y(t-\tau)d\tau \overset{CTFS}{\Leftrightarrow} Ta_k b_k$$

$$x(t)y(t) \overset{CTFS}{\Leftrightarrow} \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

$$\frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

- CTFT

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

- CTFT Properties

$$x(-t) \overset{CTFT}{\Leftrightarrow} X(-\omega)$$

$$x(t-t_0) \overset{CTFT}{\Leftrightarrow} X(\omega)e^{-j\omega t_0}$$

$$x(at) \overset{CTFT}{\Leftrightarrow} \frac{1}{|a|}X(\omega/a)$$

$$X(t) \overset{CTFT}{\Leftrightarrow} 2\pi x(-\omega)$$

$$x(t)e^{j\omega_0 t} \overset{CTFT}{\Leftrightarrow} X(\omega-\omega_0)$$

$$x(t)y(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{2\pi}X(\omega) * Y(\omega)$$

$$x(t) * y(t) \overset{CTFT}{\Leftrightarrow} X(\omega)Y(\omega)$$

$$\frac{dx(t)}{dt} \overset{CTFT}{\Leftrightarrow} j\omega X(\omega)$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

If $x(t) \overset{CTFS}{\Leftrightarrow} a_k$ then

$$x(t) \overset{CTFT}{\Leftrightarrow} \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - 2\pi k/T)$$

- CTFT pairs

$$\text{sinc}(t) \overset{CTFT}{\Leftrightarrow} \text{rect}(\omega/(2\pi))$$

$$\text{rect}(t) \overset{CTFT}{\Leftrightarrow} \text{sinc}(\omega/(2\pi))$$

For $a > 0$

$$\frac{1}{(n-1)!}t^{n-1}e^{-at}u(t) \overset{CTFT}{\Leftrightarrow} \frac{1}{(j\omega+a)^n}$$

$$\sum_{k-\infty}^{\infty} \delta(t-kT) \overset{CTFT}{\Leftrightarrow} \frac{1}{T}\sum_{k-\infty}^{\infty} 2\pi\delta(\omega-2\pi k/T)$$

- DFT

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}$$

$$x(n) = \sum_{k=0}^{N-1} X_k e^{j2\pi kn/N}$$

- DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

- DTFT pairs

$$a^n u(n) \overset{DTFT}{\Leftrightarrow} \frac{1}{1-ae^{-j\omega}}$$

- Sampling and Reconstruction

$$y(n) = x(nT)$$

$$Y(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\frac{\omega-2\pi k}{T}\right)$$

$$s(t) = \sum_{k=-\infty}^{\infty} y(k)\delta(t-kT)$$

$$S(\omega) = Y(\omega T)$$

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Problem 1.(25pt) *Formal Logic*

Consider the function $x(t)$ for $t \in \Re$.

We say that the function $x(t)$ is bounded if there exists an $M \in \Re$ such that for all $t \in \Re$, $|x(t)| \leq M$.

Alternatively, we say that a function $x(t)$ is unbounded if it is not bounded.

a) Give a formal logical expression for the definition of bounded.

b) Give a formal logical expression for the definition of unbounded.

c) Prove that the function $x(t) = e^t$ is either bounded or unbounded.

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Problem 2.(25pt) *System Properties*

For each of the following continuous-time systems with input $x(t)$ and output $y(t)$, prove or disprove that it is i) linear; ii) time-invariant; iii) BIBO stable.

a) $y(t) = e^t x(t)$

b) $y(t) = e^{x(t)}$

c) $y(t) = \int_0^t x(\tau) d\tau$

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Problem 3.(25pt) *Impulse Response*

Let $y(n) = S[x(n)]$ be an LTI discrete-time system with input $x(n)$ and output $y(n)$.

a) Prove that

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

(Hint: The easiest approach is to just apply the definition of $\delta(n)$.)

b) Prove that

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

where $h(n) \triangleq S[\delta(n)]$. Justify each step of your proof.

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Problem 4.(25pt) *Difference Equations*

Consider the discrete time system $y(n) = S[x(n)]$, which is governed by the difference equation

$$y(n) = ay(n-1) + x(n)$$

where $x(n)$ is the input and $y(n)$ is the output.

It can be proven that this is a LTI system. So for the remainder of the problem, you may assume this is true.

a) Verify that if $x(n) = \delta(n)$ then $y(n) = a^n u(n)$.

b) For what values of a is the system BIBO stable?

c) Express the output, $y(n)$, as an explicit function of the input, $x(n)$.

d) Calculate the output, $y(n)$, when $x(n) = u(n)$.

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