Uncertain Geometry for Image Analysis

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Geometric Tasks in Image Analysis
Real Images with image features
Grupping of image features

Aggregating individual features to larger entities:

- Aggregating edge pixels to edges
- Concatenation of edges
- Aggregating regions to larger ones
- Aggregating symmetric parts
Example: Aggregating edge points

Given: edge points $x_n, n = 1, \ldots, N$
Unknown: edge $(y, z)$

Method:
1. Determining fitting line $l$
2. Determining starting and end point $y \in l$ und $z \in l$
Questions:

- Do all points belong to edge?
- How accurate is line?
- How accurate are end points?
Determination of 3D-structures from image structures

Reconstruction of 3D-objects from image information

- Surfaces
- Polyhedra
- Zylinders
- 3D-Lines
Example: Forward intersection with points and lines

Observed: Image points and lines in 4 images
Given: Orientation data of images 1 to 4
Unknown: 3D-Coordinates of point
Method:
1. Determination of approximate values
2. Optimal estimation

Questions:
- Are Observations consistent?
- How accurate is result?
- What effect do errors in correspondence have?
Orientation of cameras

Determination of pose of camera at time of exposure

- Single cameras
- Multiple cameras
  - Aerial images (50 to 20000)
  - Video sequences (≥ 1500/min)
- with and without knowledge of calibration
Example: Orientation of camera from points and lines
Given: 3D-points and 3D-lines, straight line preserving mapping
Observed: Image points and lines
Unknown: Orientation and Calibration of single camera
Method:

1. Check of observations
2. Determination of approximate values
3. Optimal Estimation

Questions (as above)

▶ Are observations consistent?
▶ How accurate is the result?
▶ What effect have correspondence errors onto the result?
Types of tasks

- Determination of geometric entities
  *Intersection point, projection ray, ...*

- Check of constraints
  *Collinearity, Consistency, ...*

- Estimation of parameters
  *of lines, orientations, ...*

**under uncertainty**
Types of uncertainty

- Unavoidable random deviations can be modeled stochastically, approximately Gaussian

- Calibration errors: systematic, model errors, small deterministic or stochastic

- Occlusions: Missing points, parts of edges, parts of regions systematic, model errors, large: may be modeled stochastically

- Correspondence errors, identification errors, detection errors large, not systematic: may be modeled stochastically

- Distribution of image features may be modeled stochastically
Representation of Geometry
Geometric image features

Simple entities:

- distinct points, positions, ...
- straight image edges, -lines
- straight edge segments, line segments
Representation in homogeneous coordinates distinct points, postions, ...

\[ \mathbf{x} : \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = w \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x_0 \\ x_h \end{bmatrix} \]

with Euclidean coordinates
straight edges

$$l : \quad 1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{a^2 + b^2} \begin{bmatrix} \cos \phi \\ \sin \phi \\ -s \end{bmatrix} = \begin{bmatrix} l_h \\ l_0 \end{bmatrix}$$

with normal \([\cos \phi, \sin \phi]^T\) and distance \(s\) to origin
straight edge segment, line segment

$$s : \quad (u, v) \leftrightarrow (k, m, n)$$

starting and end point \(u\) und \(v\)
or
line \(k\) and limiting lines \(m\) und \(n\)
Simple entities:

- Distinct points, corner, nodes, ...
- Straight lines
- Planes
- Straight edge segments, line segments
Representation in homogeneous Coordinates
Distinct points, corner, nodes, ...

\[ \mathbf{X} : \quad \mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} U \\ V \\ W \\ T \end{bmatrix} = T \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} X_0 \\ X_h \end{bmatrix} \]
Planes

\[ A : \quad A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \sqrt{A^2 + B^2 + C^2} \begin{bmatrix} N \\ -S \end{bmatrix} = \begin{bmatrix} A_h \\ A_0 \end{bmatrix} \]

with normal \( N \) and distance \( S \) from origin \( x \)
straight lines: Plücker-Coordinates

\[ L : \mathbf{L} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \\ L_6 \end{bmatrix} = \begin{bmatrix} L_h \\ L_0 \end{bmatrix} \]
linear as join of two points

\[ \chi \left( \begin{bmatrix} X \\ 1 \end{bmatrix} \right) \quad \gamma \left( \begin{bmatrix} Y \\ 1 \end{bmatrix} \right) \]

\[ \mathcal{L} = \chi \wedge \gamma : \quad \mathbf{L} = \begin{bmatrix} \mathbf{Y} - \mathbf{X} \\ \mathbf{X} \times \mathbf{Y} \end{bmatrix} = \Pi(\mathbf{X}) \mathbf{Y} = -\Pi(\mathbf{Y}) \mathbf{X} \]

with \( 6 \times 4 \)-matrix \( \Pi(\mathbf{X}) \), depending on \( \mathbf{X} \)
Pi-Matrix

\[ \Pi(X) = \begin{bmatrix} X_4 & 0 & 0 & -X_1 \\ 0 & X_4 & 0 & -X_2 \\ 0 & 0 & X_4 & -X_3 \\ 0 & -X_3 & X_2 & 0 \\ X_3 & 0 & -X_1 & 0 \\ -X_2 & X_1 & 0 & 0 \end{bmatrix} \]

Plückermatrix

\[ \Gamma(L) = \begin{bmatrix} 0 & L_6 & -L_5 & -L_1 \\ -L_6 & 0 & L_4 & -L_2 \\ L_5 & -L_4 & 0 & -L_3 \\ L_1 & L_2 & L_3 & 0 \end{bmatrix} \]

\[ \Gamma(X \land Y) = X Y^T - Y X^T \]
linear as intersection of two planes

\[ L = A \cap B : \quad L = \Pi(A) B = -\Pi(B) A \]

with $6 \times 4$-matrix $\Pi(X)$, depending on $X$

\[ \Pi(A) = D_6 \Pi(A) \]

Matrix for dualling lines (exchanging first and second triplets of rows)

\[ D_6 = \begin{bmatrix} 0 & I_3 \\ I_3 & 0 \end{bmatrix}_{6 \times 6} \]

dual Plücker matrix

\[ \Gamma(L) = \begin{bmatrix} 0 & L_3 & -L_2 & -L_4 \\ -L_3 & 0 & L_1 & -L_5 \\ L_2 & -L_1 & 0 & -L_6 \\ L_4 & L_5 & L_6 & 0 \end{bmatrix} \quad \Gamma(A \cap B) = AB^T - BA^T \]
- Straight lines segments

\[ S(U, V) \leftrightarrow S(\mathcal{M}, \mathcal{B}, \mathcal{C}) \]

starting and end point \( U \) und \( V \)
or
line \( \mathcal{M} \) and limiting planes \( \mathcal{B} \) and \( \mathcal{C} \)
straight line preserving mappings: projectivities, homographies
straight line preserving planar mapping

\[
\mathbf{x}' = H \mathbf{x}
\]

8 degrees of freedom: translation (2), rotation (1), scale (1),
affinity (2), projektivity (2)
straight line preserving spatial mapping

\[
X' = H X
\]

15 degrees of freedom: translation (3), rotation (3), scale (3), affinity (3), projektivity (3)
Special properties of homogeneous entities

distance to origin:

\[ d_{xO} = \frac{|x_0|}{|x_h|} \quad d_{lO} = \frac{|l_0|}{|l_h|} \quad d_{XO} = \frac{|X_0|}{|X_h|} \quad d_{LO} = \frac{|L_0|}{|L_h|} \quad d_{AO} = \frac{|A_0|}{|A_h|} \]

entities at infinity: if homogeneous part is 0

2D:

\[ x_\infty : \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \quad l_\infty : \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

3D:

\[ X_\infty : \begin{bmatrix} X_0 \\ 0 \end{bmatrix} \quad L_\infty : \begin{bmatrix} L_0 \\ 0 \end{bmatrix} \quad A_\infty : \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]
Projection of 3D-object-point \( X \) to 2D-image points:

\[
x'_3 = P_{3\times4} X_{3\times1}
\]

with projection matrix

\[
P_{3\times4} = [p_{ij}] = \begin{bmatrix} A^T \\ B^T \\ C^T \end{bmatrix} = [p_1, p_2, p_3, p_4]
\]

- \( A \) are planes of coordinate system of camera \( S_c \)
- \( p_i \) = images of points at infinity of axes
- \( p_4 \) = image of origin
- \([p_{31}, p_{32}, p_{33}]^T\) = viewing direction
- null space = projection center
Mapping of 3D-line $\mathcal{L}$ into image line $\ell'$

$$l' = Q L$$

with projection matrix for lines

$$Q = \begin{bmatrix} M_1^T \\ M_2^T \\ M_3^T \end{bmatrix} = [q_1, q_2, q_3; q_4, q_5, q_6]$$

$M_i$ is (dual) $i$-th coordinate axis
$q_1$ to $q_3$ = images of coordinate axes
$q_4$ to $q_6$ = image so coordinate lines at infinity
$q_6$ = image of horizon!
Back projection of points and lines

projection planes

\[ A_{l'} = P^T l' \]

projection line

\[ L_{x'} = \overline{Q}^T x' \]

with \( \overline{Q} = QD_6 \)

→ geometric constructions
2D

3D
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<table>
<thead>
<tr>
<th>Construction</th>
<th>( c = U(a)b = V(b)a )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l = \chi \wedge y )</td>
<td>( l = S(x)y = -S(y)x )</td>
</tr>
<tr>
<td>( \chi = l \cap m )</td>
<td>( x = S(l)m = -S(m)l )</td>
</tr>
<tr>
<td>( L = X \wedge Y )</td>
<td>( L = \Pi(X)Y = -\Pi(Y)X )</td>
</tr>
<tr>
<td>( L = \mathcal{A} \cap B )</td>
<td>( L = \overline{\Pi}(A)B = -\overline{\Pi}(B)A )</td>
</tr>
<tr>
<td>( \mathcal{A} = L \wedge X )</td>
<td>( A = \Gamma(L)X = \overline{\Pi}^T(X)L )</td>
</tr>
<tr>
<td>( X = L \cap \mathcal{A} )</td>
<td>( X = \overline{\Gamma}(L)A = \Pi^T(A)L )</td>
</tr>
<tr>
<td>( X \rightarrow \chi' )</td>
<td>( x' = P \ X = (I_3 \otimes X^T) \ \text{vec}(P^T) )</td>
</tr>
<tr>
<td>( L \rightarrow l' )</td>
<td>( l' = Q \ L = (I_3 \otimes \overline{L}^T) \ \text{vec}(Q^T) )</td>
</tr>
<tr>
<td>( \chi' \rightarrow L \chi' )</td>
<td>( L_{x'} = \overline{Q}^T \ x' = (x'^T \otimes I_6) \ \text{vec}\overline{Q} )</td>
</tr>
<tr>
<td>( l' \rightarrow \mathcal{A}_l' )</td>
<td>( \mathcal{A}_{l'} = P^T \ l' = (l'^T \otimes I_4) \ \text{vec}P )</td>
</tr>
</tbody>
</table>
Uncertain Geometric Reasoning
Assumption:
Usefulness of homogeneous representation
Extension of representation by uncertainty
Uncertainty of Homogeneous Vectors

Principle:

- Euclidean entity \( \mathbb{R}^n \)
- Projective entity \( \mathbb{P}^n \)
- Homogeneous coordinates \( \mathbb{R}^{n+k} \)

- Euclidean entity \( \mathbb{R}^m \)
- Projective entity \( \mathbb{P}^m \)
- Homogeneous coordinates \( \mathbb{R}^{m+l} \)

- Simple, rigorous/approximate
- Difficult
- Simple, approximate
What is uncertainty of points in homogeneous coordinates?

Equivalence classes (arbitrary scaling)

\[ p(x) \equiv p(y) \quad \text{iff} \quad x = \lambda y \]

projective points in \( \mathbb{P}^n \) are straight lines through \( O \) in \( \mathbb{R}^{n+1} \)
Uncertainty of a straight line?
Uncertainty of a direction?
Uncertainty of direction in plane
v. Mises distribution, uncertainty of direction vector
Uncertain directions in $\mathbb{R}^3$
uncertain points $x$ and lines $l$ in the plane (2 d. o. f.) →

$$\begin{bmatrix} x & \Sigma_{xx} \\ 3 \times 1 & 3 \times 3 \end{bmatrix} \quad \begin{bmatrix} l & \Sigma_{ll} \\ 3 \times 1 & 3 \times 3 \end{bmatrix}$$

uncertain points $X$, lines $L$ and planes $A$ in space (3, 4, and 3 d. o. f.) →

$$\begin{bmatrix} X & \Sigma_{XX} \\ 4 \times 1 & 4 \times 4 \end{bmatrix} \quad \begin{bmatrix} L & \Sigma_{LL} \\ 6 \times 1 & 6 \times 6 \end{bmatrix} \quad \begin{bmatrix} A & \Sigma_{AA} \\ 4 \times 1 & 4 \times 4 \end{bmatrix}$$

uncertain projection parameters (11 d. o. f.)

$$\begin{bmatrix} p & \Sigma_{pp} \\ 12 \times 1 & 12 \times 12 \end{bmatrix} \quad \begin{bmatrix} q & \Sigma_{qq} \\ 18 \times 1 & 18 \times 18 \end{bmatrix}$$
uncertain construction (bilinear)

\[ c = U(a)b = V(b)a \]

then

\[ \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{bmatrix} \rightarrow [c, \Sigma_{cc}] \]

\[ \Sigma_{cc} = U(a)\Sigma_{bb}U^T(a) + V(b)\Sigma_{ab}U^T(a) + U(a)\Sigma_{ba}V^T(b) + V(b)\Sigma_{aa}V^T(b) \]

simple error propagation independent on distribution

Degree of approximation: relative bias in \( \mu \) and \( \sigma^2 \) = directional uncertainty
Testing Geometric Relations

Example: Testing Identity of Two 2D-points

Test of \( x = y \)

**Classical procedure**

Difference:

\[
d = y - x \sim N(\mu_d, \Sigma_{dd}) = N(\mu_y - \mu_x, \Sigma_{xx} + \Sigma_{yy})
\]

Test of

\[
H_0 : \mu_d = 0 \quad H_a : \mu_d \neq 0
\]

Test statistic

\[
T = d^T \Sigma_{dd}^{-1} d \sim \chi^2_2
\]

**Problem:** too complex for general geometric relations
General procedure
'Difference': line 1 generated by x and y is not defined, thus $1 = 0$

\[ \mathbf{d}|_{H_0} = \mathbf{x} \times \mathbf{y}|_{H_0} \sim N(\mathbf{0}, \Sigma_{dd}) \]

\[ \Sigma_{dd} = S(\mu_x) \Sigma_{yy} S^T(\mu_x) + S(\mu_y) \Sigma_{xx} S^T(\mu_y) \]

Problems:
- $\mu_x$ and $\mu_y$ not known
- number of elements in $\mathbf{d}$ too large, depending on constraints

Solution:
+ Use $\hat{\mu}_x = x$ and $\hat{\mu}_y = y$ as approximations
+ Select independent constraints (cf. above)
Discussion:
+ simple
+ fast
+ very good approximation if test is not rejected
+ approximate test statistic increases monotonically with rigorous one

0 Conditioning and Normalization necessary to reduce bias
— only approximation if test is rejected
Normalization only of covariance matrix, no scaling necessary
1. determine the difference $d$, $d$, $D$ or $D$ (cf. tables 3, 2).
2. select $r$ independent constraints
3. determine the covariance matrix $\Sigma_{dd}$ of the $r$ selected elements $d$ of differences
4. determine the test statistic $T$

$$T = d^T \Sigma_{dd}^+ d \sim \chi^2_r$$

5. choose a significance number $\alpha$
   compare $T$ with the critical value $\chi^2_{r,\alpha}$.
   If $T > \chi^2_{r,\alpha}$ then reject hypothesis on relation
<table>
<thead>
<tr>
<th>No.</th>
<th>2D-entities</th>
<th>relation</th>
<th>dof</th>
<th>test</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x, y )</td>
<td>( x \equiv y )</td>
<td>2</td>
<td>( d = S(x)y = -S(y)x )</td>
</tr>
<tr>
<td>2</td>
<td>( x, l )</td>
<td>( x \in l )</td>
<td>1</td>
<td>( d = x^Tl = l^Tx )</td>
</tr>
<tr>
<td>3</td>
<td>( l, m )</td>
<td>( l \equiv m )</td>
<td>2</td>
<td>( d = S(l)m = -S(m)l )</td>
</tr>
</tbody>
</table>

Tabelle: shows 3 relationships between points and lines useful for 2D grouping, together with the degree of freedom and the essential part of the test statistic.
<table>
<thead>
<tr>
<th>No.</th>
<th>3D-entities</th>
<th>relation</th>
<th>dof</th>
<th>test</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$X, \mathcal{Y}$</td>
<td>$X \equiv \mathcal{Y}$</td>
<td>3</td>
<td>$D = \Pi(X)Y = -\Pi(Y)X$</td>
</tr>
<tr>
<td>5</td>
<td>$X, L$</td>
<td>$X \in L$</td>
<td>2</td>
<td>$D = \overline{\Pi}^T(X)L = \overline{\Gamma}^T(L)X$</td>
</tr>
<tr>
<td>6</td>
<td>$X, A$</td>
<td>$X \in A$</td>
<td>1</td>
<td>$d = X^TA = A^TX$</td>
</tr>
<tr>
<td>7</td>
<td>$L, M$</td>
<td>$L \equiv M$</td>
<td>4</td>
<td>$D = \Gamma(L)\Gamma(M)$</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>$L \cap M \neq \emptyset$</td>
<td>1</td>
<td>$d = L^TM = M^TL$</td>
</tr>
<tr>
<td>9</td>
<td>$L, A$</td>
<td>$L \in A$</td>
<td>2</td>
<td>$D = \Pi^T(A)L = \Gamma^T(L)A$</td>
</tr>
<tr>
<td>10</td>
<td>$A, B$</td>
<td>$A \equiv B$</td>
<td>3</td>
<td>$D = \Pi(A)B = -\Pi(B)A$</td>
</tr>
</tbody>
</table>

Tabelle: shows 7 relationships between points, lines and planes useful for 3D grouping, together with the degree of freedom and the essential part of the test statistic.
<table>
<thead>
<tr>
<th>No.</th>
<th>entities</th>
<th>relation</th>
<th>dof</th>
<th>test</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x'$, $P(P)$, $X$</td>
<td>$x' \equiv P(X)$</td>
<td>2</td>
<td>$d = S(x')PX = 0$</td>
</tr>
<tr>
<td>2</td>
<td>$\ell'$, $P(P)$, $L$</td>
<td>$\ell' \equiv P(L)$</td>
<td>2</td>
<td>$D = \Gamma(L)P^T\ell' = 0$</td>
</tr>
<tr>
<td>3</td>
<td>$X$, $Y$, $Z$, $T$</td>
<td>coplanar</td>
<td>1</td>
<td>$d =</td>
</tr>
<tr>
<td>4</td>
<td>$A$, $B$, $C$, $D$</td>
<td>intersect</td>
<td>1</td>
<td>$d =</td>
</tr>
</tbody>
</table>

Tabelle: *shows 4 multi linear relationships together with the degree of freedom and the essential part of the test statistic.*
Grouping

Intermediate step:
Given: edge segment $s(x, y)$, point $z$
Unknown: Does $z \in s$ hold?
Tests with three lines \((l, m, n)\):

\[
z^T l = 0 \quad \text{sign} \left( \frac{z^T m}{|m_0|} \right) \neq \text{sign} \left( \frac{z^T n}{|n_0|} \right)
\]
Combined estimation of a 3D-line

\[
\begin{bmatrix}
  L_{11}'^T \\
  \Pi^T(A_{52}') \\
  L_{23}'^T \\
  \Pi^T(A_{54}')
\end{bmatrix}
\begin{bmatrix}
  x'_{11}^TQ_1 \\
  \Pi^T(P_{21}'_{52}) \\
  x'_{23}^TQ_3 \\
  \Pi^T(P_{41}'_{54})
\end{bmatrix}
\]

of \( A \rightarrow \hat{L}_5 \)

\( L_5 = A\hat{L}_5 = w = 0 \) SVD
Results and Outlook

Result
- Integration of geometry and uncertainty
- Homogeneous representation suited
- Software SUGR (statistically uncertain geometric reasoning) in JAVA available

Use
- Grouping of image and space features
- Reconstruction from images
- Reconstruction from laser range data

Open problems
- Limitations of approach
- Quality of reasoning for ling chains
- Integration of other types of uncertainties (Correspondence, grouping, ...)

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How can we determine the covariance matrix between different entities?

Example: Given three 3D-points \(X, Y\) and \(Z\), and a plane \(A\):

1. Determine lines

\[
L = X \wedge Y \quad M = X \wedge Z
\]

2. Determine fourth point

\[
T = L \cap A
\]

3. Determine plane

\[
B = M \wedge T
\]

Plane \(M \wedge T\) should be identical to plane \(X \wedge Y \wedge Z\).

Line \(M\) and point \(T\) both depend on \(X\): \(\Sigma_{MT} \neq 0\).

If covaraince \(\Sigma_{MT}\) is neglected, then \(D(M \wedge T) \neq D(X \wedge Y \wedge Z)\).
Construction of plane $\mathbf{B} = \mathbf{M} \wedge \mathbf{T}$ with $\mathbf{M} = \mathbf{X} \wedge \mathbf{Z}$ and $\mathbf{T} = \mathbf{A} \cap (\mathbf{X} \wedge \mathbf{Y})$
General setup:

Given:

– mutually independent vectors \((x, \Sigma_{xx}), (y, \Sigma_{yy})\) and \((z, \Sigma_{zz})\)

– linear functions

\[
\begin{align*}
    u &= Ax + Bb \\
    v &= Cx + Dc
\end{align*}
\]

The covariance matrix of \(u\) and \(v\) is given by:

\[
\Sigma_{uv} = A \Sigma_{xx} C^T
\]
Proof:

from

\[ z = Et \]

with

\[
t = \begin{bmatrix}
    a \\
    b \\
    c
\end{bmatrix}
\]
\[
E = \begin{bmatrix}
    A & B & 0 \\
    C & 0 & D
\end{bmatrix}
\]
\[
z = \begin{bmatrix}
    u \\
    v
\end{bmatrix}
\]

we obtain

\[ \Sigma_{zz} = E \Sigma_{tt} E^T \]

with

\[
\Sigma_{zz} = \begin{bmatrix}
    \Sigma_{uu} & \Sigma_{uv} \\
    \Sigma_{vu} & \Sigma_{vv}
\end{bmatrix} = \begin{bmatrix}
    A & B & 0 \\
    C & 0 & D
\end{bmatrix} \begin{bmatrix}
    \Sigma_{xx} & 0 & 0 \\
    0 & \Sigma_{yy} & 0 \\
    0 & 0 & \Sigma_{zz}
\end{bmatrix} \begin{bmatrix}
    A^T \\
    B^T \\
    C^T
\end{bmatrix}
\]