Uncertain Geometry for Image Analysis

Wolfgang Förstner

University of Bonn

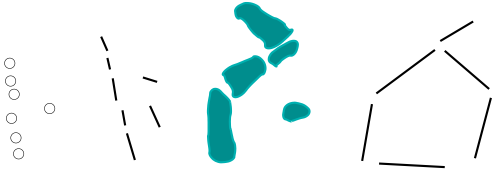
Purdue, 20. March 2008

Geometric Tasks in Image Analysis

Real Images with image features

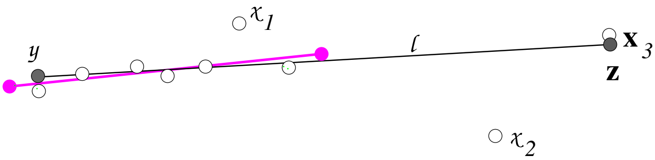
Aggregating individual features to larger entie:

- Aggregating edge pixels to edges
- Concatenation of edges
- aggregating regions to larger ones
- aggregating symmetic parts



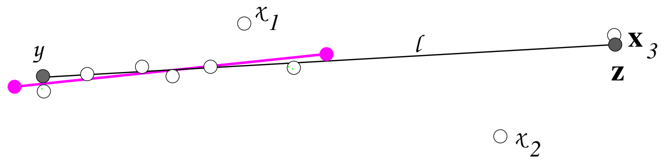
Example: Aggregating edge points

Given: edge points $x_n, n = 1, ..., N$
Unknown: edge (y, z)



Method:

1. Determining fitting line l
2. Determining starting and end point $y \in l$ und $z \in l$

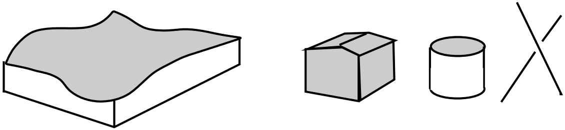


Quastions:

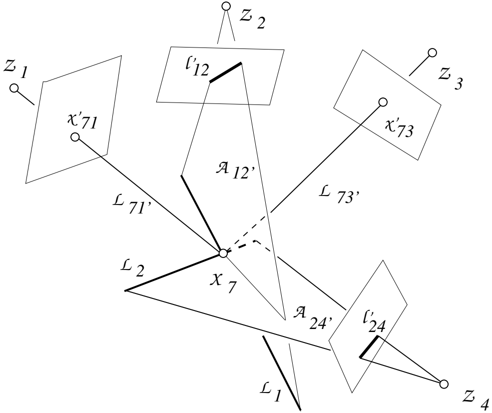
- Do all points belong to edge?
- How accurate is line?
- How accurate are end points?

Reconstruction of 3D-objects from image information

- Surfaces
- Polyhedra
- Zylinders
- 3D-Lines



Example: Forward intersection with points and lines



Observed: Image points and lines in 4 images
Given: Orientation data of images 1 to 4
Unknown: 3D-Coordinates of point

Method:

1. Determination of approximate values
2. Optimal estimation

Quastions:

- Are Observations consistent?
- How accurate is result?
- What effect do errors in correspondence have?

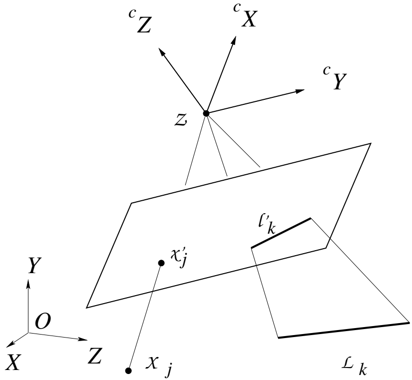
Orientation of cameras

Determination of pose of camera at time of exposure

- Single cameras
- Multiple cameras
  - Aerial images (50 to 20000)
  - Video sequences ($\geq$ 1500/min)
- with and without knowledge of calibration

Example: Orientation of camera from points and lines

Given: 3D-points and 3D-lines, streight line preserving mapping
Observed: Image points and lines
Unknown: Orientation and Calibration of single camera



Method:

1. Check of observations
2. Determination of approximate values
3. Optimal Estimation

Questions (as above)

- Are observations consistent?
- How accurate is the result?
- What effect have correspondence errors onto the result?

- Determination of geometric entities
  Intersection point, projection ray, ...
- Check of constraints
  Collinearity, Consistency, ...
- Estimation of parameters
  of lines, orientations, ...

under uncertainty

Types of uncertainty

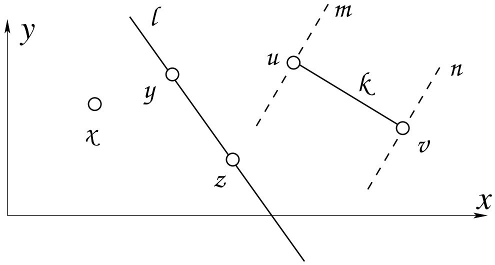
- Unavoidable random deviations
  can be modeled stochastically, approximately Gaussian
- Calibration errors: systematic, model errors, small
  deterministic or stochastic
- Occlusions: Missing points, parts of edges, pars of regions
  systematic, model errors, large: *may be modeled stochastically*

- Correspondence errors, identification errors, detection errors
  glarge, not systematic: *may be modeled stochastically*
- Distribution of image features
  may be modeled stochastically

Representation of Geometry

Simple entities:

- distinct points, positions , ...
- straight image edges, -lines
- straight edge segments, line segments



Representation in homogeneous coordinates distinct points, postions, ...

$$\chi : \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = w \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_0 \\ x_h \end{bmatrix}$$

with Euclidean coordinates

straight edges

$$\mathcal{l}: \quad \mathbf{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{a^2+b^2} \begin{bmatrix} \cos\phi \\ \sin\phi \\ -s \end{bmatrix} = \begin{bmatrix} \boldsymbol{l}_h \\ l_0 \end{bmatrix}$$

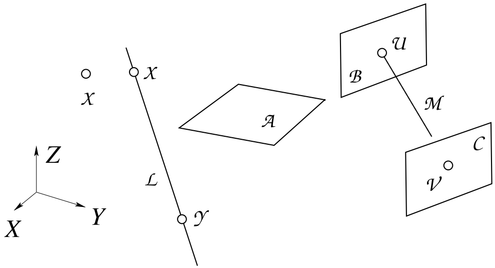
with normal $[\cos\phi, \sin\phi]^\mathsf{T}$ and distance s to origin
straight edge segment, line segment

$$\mathcal{s}: \quad (\mathcal{u}, \mathcal{v}) \leftrightarrow (\mathcal{k}, \mathcal{m}, \mathcal{n})$$

starting and end point $\mathcal{u}$ und $\mathcal{v}$
or
line $\mathcal{k}$ and limiting lines $\mathcal{m}$ und $\mathcal{n}$

Simple entities:

- Distinct points, corner, nodes, ...
- Straight lines
- Planes
- Straight edge segments, line segments



Representation in homogeneous Coordinates
Distinct points, corner, nodes, ...

$$\mathcal{X}: \quad \mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} U \\ V \\ W \\ T \end{bmatrix} = T \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}_0 \\ X_h \end{bmatrix}$$

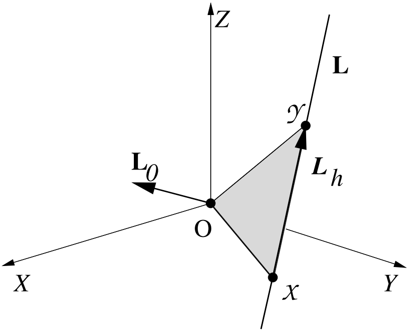
Planes

$$\mathcal{A}: \quad \mathbf{A} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \sqrt{A^2 + B^2 + C^2} \begin{bmatrix} \boldsymbol{N} \\ -S \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_h \\ A_0 \end{bmatrix}$$

with normal $\boldsymbol{N}$ and distance S from origin x

straight lines: Plücker-Coordinates

$$\mathcal{L}: \quad \underset{6\times 1}{\mathbf{L}} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \\ L_6 \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}_h \\ \boldsymbol{L}_0 \end{bmatrix}$$



linear as join of two points

$$\mathcal{X}\left(\begin{bmatrix} \boldsymbol{X} \\ 1 \end{bmatrix}\right) \qquad \mathcal{Y}\left(\begin{bmatrix} \boldsymbol{Y} \\ 1 \end{bmatrix}\right)$$

$$\mathcal{L} = \mathcal{X} \wedge \mathcal{Y}: \quad \underset{6\times1}{\mathbf{L}} = \begin{bmatrix} \boldsymbol{Y} - \boldsymbol{X} \\ \boldsymbol{X} \times \boldsymbol{Y} \end{bmatrix} = \underset{6\times4}{\Pi(\mathbf{X})} \underset{4\times1}{\mathbf{Y}} = -\underset{6\times4}{\Pi(\mathbf{Y})} \underset{4\times1}{\mathbf{X}}$$

with 6×4-matrix $\Pi(\mathbf{X})$, depending on $\mathbf{X}$

Pi-Matrix

$$\Pi(\mathbf{X}) = \begin{bmatrix} X_4 & 0 & 0 & -X_1 \\ 0 & X_4 & 0 & -X_2 \\ 0 & 0 & X_4 & -X_3 \\ 0 & -X_3 & X_2 & 0 \\ X_3 & 0 & -X_1 & 0 \\ -X_2 & X_1 & 0 & 0 \end{bmatrix}$$

Plückermatrix

$$\Gamma(\mathbf{L}) = \begin{bmatrix} 0 & L_6 & -L_5 & -L_1 \\ -L_6 & 0 & L_4 & -L_2 \\ L_5 & -L_4 & 0 & -L_3 \\ L_1 & L_2 & L_3 & 0 \end{bmatrix} \qquad \Gamma(\mathcal{X} \wedge \mathcal{Y}) = \mathbf{X}\mathbf{Y}^{\mathsf{T}} - \mathbf{Y}\mathbf{X}^{\mathsf{T}}$$

linear as intersection of two planes

$$\mathcal{L} = \mathcal{A} \cap \mathcal{B}: \quad \mathbf{L} = \overline{\mathsf{\Pi}}(\mathbf{A})\,\mathbf{B} = -\overline{\mathsf{\Pi}}(\mathbf{B})\,\mathbf{A}$$

with 6×4-matrix $\overline{\mathsf{\Pi}}(\mathbf{X})$, depending on $\mathbf{X}$

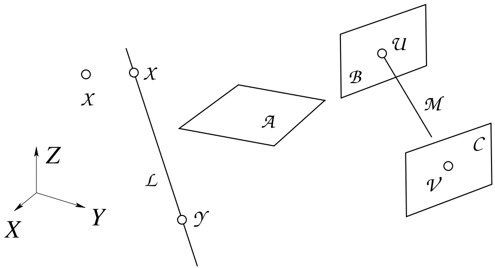
$$\overline{\mathsf{\Pi}}(\mathbf{A}) = D_6\,\mathsf{\Pi}(\mathbf{A})$$

Matrix for dualling lines (exchanging first and second triplets of rows)

$$D_6 = \underset{6\times 6}{\begin{bmatrix} \mathbf{0} & I_3 \\ I_3 & \mathbf{0} \end{bmatrix}}$$

dual Plücker matrix

$$\overline{\Gamma}(\mathbf{L}) = \begin{bmatrix} 0 & L_3 & -L_2 & -L_4 \\ -L_3 & 0 & L_1 & -L_5 \\ L_2 & -L_1 & 0 & -L_6 \\ L_4 & L_5 & L_6 & 0 \end{bmatrix} \qquad \overline{\Gamma}(\mathcal{A} \cap \mathcal{B}) = \mathbf{A}\mathbf{B}^{\mathsf{T}} - \mathbf{B}\mathbf{A}^{\mathsf{T}}$$



- Straight lines segments

$$\mathcal{S}(\mathcal{U}, \mathcal{V}) \leftrightarrow \mathcal{S}\left(\mathcal{M}, \mathcal{B}, \mathcal{C}\right)$$

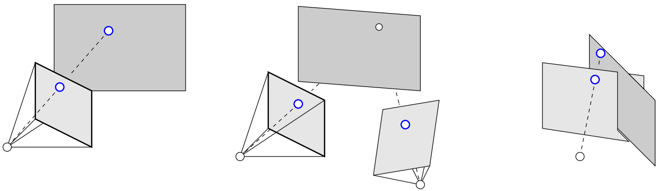
starting and end point $\mathcal{U}$ und $\mathcal{V}$
or
line $\mathcal{M}$ and limiting planes $\mathcal{B}$ and $\mathcal{C}$

straight line preserving mappings: projectivities, homographies

straight line preserving planar mapping

$$\underset{3\times 1}{\mathbf{x}'} = \underset{3\times 3}{\mathsf{H}} \; \underset{3\times 1}{\mathbf{x}}$$

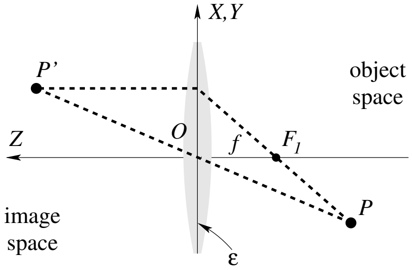
8 degrees of freedom: translation (2), rotation (1), scale (1), affinity (2), projektivity (2)



straight line preserving spatial mapping

$$\underset{4\times 1}{\mathbf{X}'} = \underset{4\times 4}{\mathsf{H}} \underset{4\times 1}{\mathbf{X}}$$

15 degrees of freedom: translation (3), rotation (3), scale (3), affinity (3), projektivity (3)



distance to origin:

$$d_{xO} = \frac{|\boldsymbol{x}_0|}{|x_h|} \quad d_{lO} = \frac{|l_0|}{|\boldsymbol{l}_h|} \quad d_{XO} = \frac{|\boldsymbol{X}_0|}{|X_h|} \quad d_{LO} = \frac{|\boldsymbol{L}_0|}{|\boldsymbol{L}_h|} \quad d_{AO} = \frac{|\boldsymbol{A}_0|}{|A_h|}$$

entities at infinity: if homogeneous part is 0

2D:

$$\mathcal{X}_\infty : \quad \begin{bmatrix} \boldsymbol{x}_0 \\ 0 \end{bmatrix} \quad \mathcal{L}_\infty : \quad \begin{bmatrix} \boldsymbol{0} \\ 1 \end{bmatrix}$$

3D:

$$\mathcal{X}_\infty : \quad \begin{bmatrix} \boldsymbol{X}_0 \\ 0 \end{bmatrix} \quad \mathcal{L}_\infty : \quad \begin{bmatrix} \boldsymbol{L}_0 \\ \boldsymbol{0} \end{bmatrix} \quad \mathcal{A}_\infty : \quad \begin{bmatrix} \boldsymbol{0} \\ 1 \end{bmatrix}$$

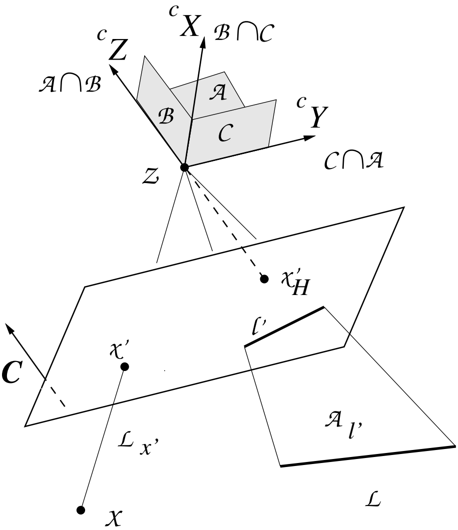
Projection of 3D-object-point $\mathcal{X}$ to 2D-image points:

$$\underset{3\times 1}{\mathbf{x}'} = \underset{3\times 4}{\mathsf{P}}\ \underset{3\times 1}{\mathbf{X}}$$

with projection matrix

$$\underset{3\times 4}{\mathsf{P}} = [p_{ij}] = \begin{bmatrix} \mathbf{A}^{\mathsf{T}} \\ \mathbf{B}^{\mathsf{T}} \\ \mathbf{C}^{\mathsf{T}} \end{bmatrix} = [\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4]$$

$\mathbf{A}$ are planes of coordinate system of camera S_c
$\mathbf{p}_i$ = images of points at infinity of axes
$\mathbf{p}_4$ = image of origin
$[p_{31}, p_{32}, p_{33}]^{\mathsf{T}}$ = viewing direction
null space = projection center



${}^cX$
$\mathcal{B} \cap \mathcal{C}$
${}^cZ$
$\mathcal{A} \cap \mathcal{B}$
$\mathcal{A}$
$\mathcal{B}$
$\mathcal{C}$
${}^cY$
$\mathcal{C} \cap \mathcal{A}$
$\mathcal{Z}$
$\mathcal{x}'_H$
l'
$\mathcal{x}'$
$\boldsymbol{C}$
$\mathcal{A}_{l'}$
$\mathcal{L}_{x'}$
$\mathcal{X}$
$\mathcal{L}$

Mapping of 3D-line $\mathcal{L}$ into image line ℓ'

$$\underset{3\times1}{\mathbf{l}'} = \underset{3\times6}{\mathsf{Q}} \underset{6\times1}{\mathbf{L}}$$

with projection matrix for lines

$$\mathsf{Q} = \begin{bmatrix} \mathbf{M}_1^{\mathsf{T}} \\ \mathbf{M}_2^{\mathsf{T}} \\ \mathbf{M}_3^{\mathsf{T}} \end{bmatrix} = [\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3;\ \mathbf{q}_4, \mathbf{q}_5, \mathbf{q}_6]$$

$\mathbf{M}_i$ is (dual) i-th coordinate axis
$\mathbf{q}_1$ to $\mathbf{q}_3$ = images of coordinate axes
$\mathbf{q}_4$ to $\mathbf{q}_6$ = image so coordinate lines at infinity
$\mathbf{q}_6$ = image of *horizon*!

Back projection of points and lines

projection planes

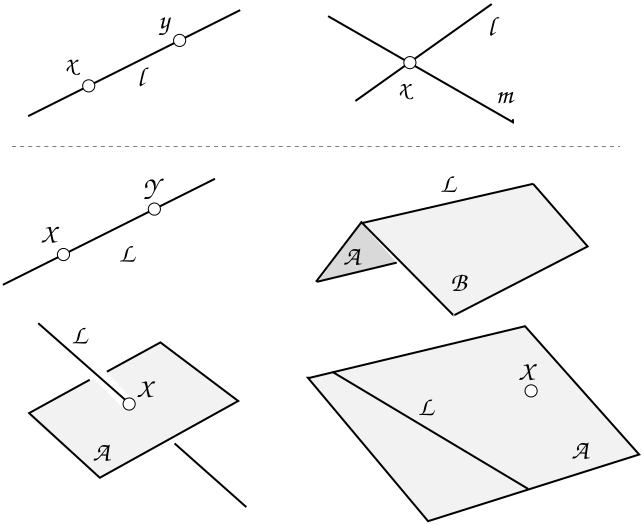
$$\underset{4\times 1}{\mathbf{A}_{l'}} = \underset{4\times 3}{\mathsf{P}^{\mathsf{T}}} \; \underset{3\times 1}{\mathbf{l}'}$$

projection line

$$\underset{6\times 1}{\mathbf{L}_{x'}} = \underset{6\times 3}{\overline{\mathsf{Q}}^{\mathsf{T}}} \; \underset{3\times 1}{\mathbf{x}'}$$

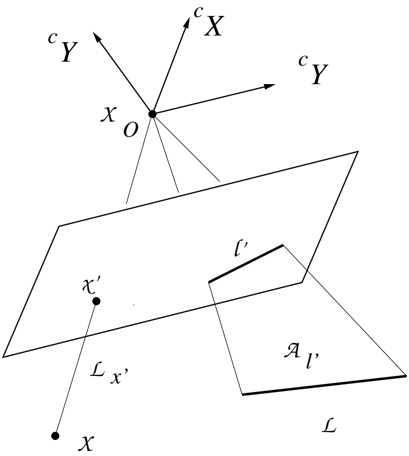
with $\overline{\mathsf{Q}} = \mathsf{Q} D_6$

$\rightarrow$ **geometric constructions**



2D

3D



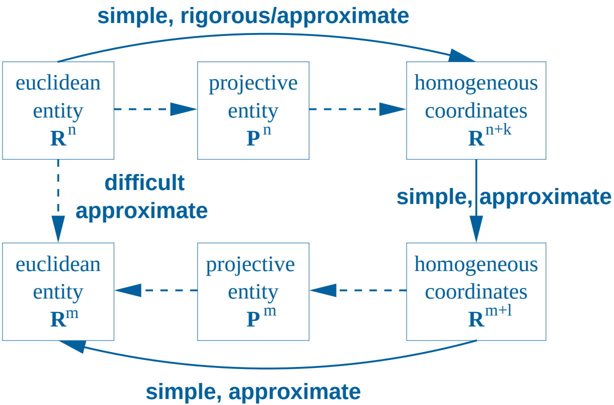
cY
cX
cY
X O
l'
x'
A l'
L x'
L
X

Construction	$\mathbf{c} = \mathsf{U}(\mathbf{a})\mathbf{b} = \mathsf{V}(\mathbf{b})\mathbf{a}$
$\mathcal{l} = \mathcal{x} \wedge \mathcal{y}$	$\mathbf{l} = \mathsf{S}(\mathbf{x})\mathbf{y} = -\mathsf{S}(\mathbf{y})\mathbf{x}$
$\mathcal{x} = \mathcal{l} \cap \mathcal{m}$	$\mathbf{x} = \mathsf{S}(\mathbf{l})\mathbf{m} = -\mathsf{S}(\mathbf{m})\mathbf{l}$
$\mathcal{L} = \mathcal{X} \wedge \mathcal{Y}$	$\mathbf{L} = \Pi(\mathbf{X})\mathbf{Y} = -\Pi(\mathbf{Y})\mathbf{X}$
$\mathcal{L} = \mathcal{A} \cap \mathcal{B}$	$\mathbf{L} = \overline{\Pi}(\mathbf{A})\mathbf{B} = -\overline{\Pi}(\mathbf{B})\mathbf{A}$
$\mathcal{A} = \mathcal{L} \wedge \mathcal{X}$	$\mathbf{A} = \Gamma(\mathbf{L})\mathbf{X} = \overline{\Pi}^{\mathsf{T}}(\mathbf{X})\mathbf{L}$
$\mathcal{X} = \mathcal{L} \cap \mathcal{A}$	$\mathbf{X} = \overline{\Gamma}(\mathbf{L})\mathbf{A} = \Pi^{\mathsf{T}}(\mathbf{A})\mathbf{L}$
$\mathcal{X} \underset{\mathcal{P}}{\rightarrow} \mathcal{x}'$	$\mathbf{x}' = \mathsf{P}\ \mathbf{X} = (\mathsf{I}_3 \otimes \mathbf{X}^{\mathsf{T}})\ \mathsf{vec}(\mathsf{P}^{\mathsf{T}})$
$\mathcal{L} \underset{\mathcal{P}}{\rightarrow} \mathcal{l}'$	$\mathbf{l}' = \mathsf{Q}\ \mathbf{L} = (\mathsf{I}_3 \otimes \overline{\mathbf{L}}^{\mathsf{T}})\ \mathsf{vec}(\mathsf{Q}^{\mathsf{T}})$
$\mathcal{x}' \underset{\mathcal{P}+}{\rightarrow} \mathcal{L}_{\mathcal{x}'}$	$\mathbf{L}_{x'} = \overline{\mathsf{Q}}^{\mathsf{T}}\ \mathbf{x}' = (\mathbf{x}'^{\mathsf{T}} \otimes \mathsf{I}_6)\ \mathsf{vec}\overline{\mathsf{Q}}$
$\mathcal{l}' \underset{\mathcal{P}+}{\rightarrow} \mathcal{A}_{\mathcal{l}'}$	$\mathbf{A}_{l'} = \mathsf{P}^{\mathsf{T}}\ \mathbf{l}' = (\mathbf{l}'^{\mathsf{T}} \otimes \mathsf{I}_4)\ \mathsf{vec}\mathsf{P}$

Uncertain Geometric Reasoning

Assumption:
Usefulness of homogeneous representation
Extension of representation by uncertainty

Principle:


simple, rigorous/approximate
euclidean entity $\mathbf{R}^n$
projective entity $\mathbf{P}^n$
homogeneous coordinates $\mathbf{R}^{n+k}$
difficult approximate
simple, approximate
euclidean entity $\mathbf{R}^m$
projective entity $\mathbf{P}^m$
homogeneous coordinates $\mathbf{R}^{m+l}$
simple, approximate

What is uncertainty of points in homogeneous coordinates?

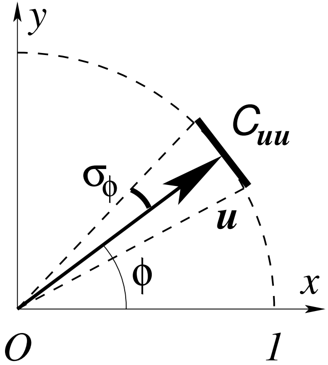
Equivalence classes (arbitrary scaling)

$$p(\mathbf{x}) \equiv p(\mathbf{y}) \quad \text{iff } \mathbf{x} = \lambda \mathbf{y}$$

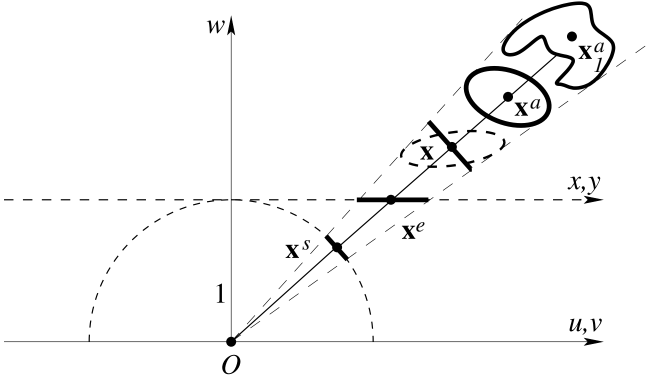
projective points in $\mathbb{P}^{\mathrm{n}}$ are straight lines through O in $\mathbb{R}^{\mathrm{n}+1}$

Uncertainty of a straight line?
Uncertainty of a direction?

Uncertainty of direction in plane
v. Mises distribution, uncertainty of direction vector



Uncertain directions in $\mathbb{R}^3$



uncertain points $\mathbf{x}$ and lines $\mathbf{l}$ in the plane (2 d. o. f.) $\rightarrow$

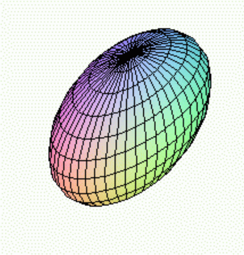
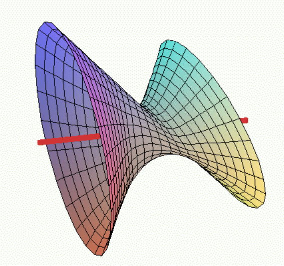
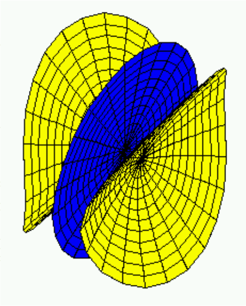
$$[\underset{3\times 1}{\mathbf{x}}, \underset{3\times 3}{\Sigma_{xx}}] \qquad [\underset{3\times 1}{\mathbf{l}}, \underset{3\times 3}{\Sigma_{ll}}]$$

uncertain points $\mathbf{X}$, lines $\mathbf{L}$ and planes $\mathbf{A}$ in space (3, 4, and 3 d. o. f.) $\rightarrow$

$$[\underset{4\times 1}{\mathbf{X}}, \underset{4\times 4}{\Sigma_{XX}}] \qquad [\underset{6\times 1}{\mathbf{L}}, \underset{6\times 6}{\Sigma_{LL}}] \qquad [\underset{4\times 1}{\mathbf{A}}, \underset{4\times 4}{\Sigma_{AA}}]$$

uncertain projection parameters (11 d. o. f.)

$$[\underset{12\times 1}{\mathbf{p}}, \underset{12\times 12}{\Sigma_{pp}}] \qquad [\underset{18\times 1}{\mathbf{q}}, \underset{18\times 18}{\Sigma_{qq}}]$$

45

uncertain construction (bilinear)

$$\underline{\mathbf{c}} = \mathsf{U}(\underline{\mathbf{a}})\underline{\mathbf{b}} = \mathsf{V}(\underline{\mathbf{b}})\underline{\mathbf{a}}$$

then

$$\left[\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \begin{bmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{bmatrix}\right] \rightarrow [\mathbf{c}, \Sigma_{cc}]$$

$$\Sigma_{cc} = \mathsf{U}(\mathbf{a})\Sigma_{bb}\mathsf{U}^\mathsf{T}(\mathbf{a}) + \mathsf{V}(\mathbf{b})\Sigma_{ab}\mathsf{U}^\mathsf{T}(\mathbf{a}) + \mathsf{U}(\mathbf{a})\Sigma_{ba}\mathsf{V}^\mathsf{T}(\mathbf{b}) + \mathsf{V}(\mathbf{b})\Sigma_{aa}\mathsf{V}^\mathsf{T}(\mathbf{b})$$

simple error propagation independent on distribution

Degree of approximation: relative bias in μ and σ^2 = directional uncertainty

Example: Testing Identity of Two 2D-points

Test of $\boldsymbol{x} = \boldsymbol{y}$

Classical procedure

Difference:

$$\boldsymbol{d} = \boldsymbol{y} - \boldsymbol{x} \sim N(\boldsymbol{\mu}_d, \Sigma_{dd}) = N(\boldsymbol{\mu}_y - \boldsymbol{\mu}_x, \Sigma_{xx} + \Sigma_{yy})$$

Test of

$$H_0 : \ \boldsymbol{\mu}_d = \boldsymbol{0} \qquad H_a : \ \boldsymbol{\mu}_d \neq \boldsymbol{0}$$

Test statistic

$$T = \boldsymbol{d}^{\mathsf{T}} \Sigma_{dd}^{-1} \boldsymbol{d} \sim \chi_2^2$$

Problem: too complex for general geometric relations

General procedure

'Difference': line $\mathbf{l}$ generated by $\mathbf{x}$ and $\mathbf{y}$ is not defined, thus $\mathbf{l} = \mathbf{0}$

$$\mathbf{d}|H_0 = \mathbf{x} \times \mathbf{y}|H_0 \sim N(\mathbf{0}, \Sigma_{\mathsf{dd}})$$

$$\Sigma_{\mathsf{dd}} = \mathsf{S}(\boldsymbol{\mu}_x)\Sigma_{\mathsf{yy}}\mathsf{S}^{\mathsf{T}}(\boldsymbol{\mu}_x) + \mathsf{S}(\boldsymbol{\mu}_y)\Sigma_{\mathsf{xx}}\mathsf{S}^{\mathsf{T}}(\boldsymbol{\mu}_y)$$

Problems:

– $\boldsymbol{\mu}_x$ and $\boldsymbol{\mu}_y$ not known

– number of elements in $\mathbf{d}$ too large, depending on constraints

Solution:

\+ Use $\widehat{\boldsymbol{\mu}}_x = \mathbf{x}$ and $\widehat{\boldsymbol{\mu}}_y = \mathbf{y}$ as *approximations*

\+ Select independent constraints (cf. above)

Discussion:

+ simple

+ fast

+ very good approximation if test is not rejected

+ approximate test statistic increases monotonically with rigorous one

0 Conditioning and Normalization necessary to reduce bias

— only approximation if test is rejected

Normalization only of covariance matrix, no scaling necessary

Procedure for Testing Geometric Entities

1. determine the difference d, $\mathbf{d}$, $\mathbf{D}$ or D (cf. tables 3, 2).
2. select r independent constraints
3. determine the covariance matrix Σ_{dd} of the r selected elements $\boldsymbol{d}$ of differences
4. determine the test statistic T

$$T = \boldsymbol{d}^\mathsf{T} \Sigma_{dd}^{+} \boldsymbol{d} \sim \chi_r^2$$

5. choose a significance number α
   compare T with the critical value $\chi^2_{r,\alpha}$.
   If $T > \chi^2_{r,\alpha}$ then reject hypothesis on relation

1	2	3	4	5
No.	2D-entities	relation	dof	test
1	$\mathcal{x}$, $\mathcal{y}$	$\mathcal{x} \equiv \mathcal{y}$	2	$\mathbf{d} = \mathsf{S}(\mathbf{x})\mathbf{y} = -\mathsf{S}(\mathbf{y})\mathbf{x}$
2	$\mathcal{x}$, $\mathcal{l}$	$\mathcal{x} \in \mathcal{l}$	1	$d = \mathbf{x}^\mathsf{T}\mathbf{l} = \mathbf{l}^\mathsf{T}\mathbf{x}$
3	$\mathcal{l}$, $\mathcal{m}$	$\mathcal{l} \equiv \mathcal{m}$	2	$\mathbf{d} = \mathsf{S}(\mathbf{l})\mathbf{m} = -\mathsf{S}(\mathbf{m})\mathbf{l}$

Tabelle: *shows 3 relationships between points and lines useful for 2D grouping, together with the degree of freedom and the essential part of the test statistic.*

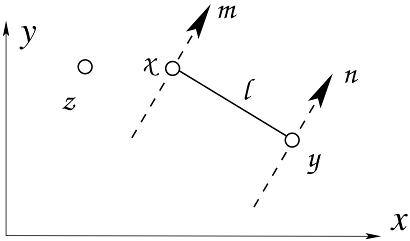
1	2	3	4	5
No.	3D-entities	relation	dof	test
4	$\mathcal{X}, \mathcal{Y}$	$\mathcal{X} \equiv \mathcal{Y}$	3	$\mathbf{D} = \Pi(\mathbf{X})\mathbf{Y} = -\Pi(\mathbf{Y})\mathbf{X}$
5	$\mathcal{X}, \mathcal{L}$	$\mathcal{X} \in \mathcal{L}$	2	$\mathbf{D} = \overline{\Pi}^\mathsf{T}(\mathbf{X})\mathbf{L} = \overline{\Gamma}^\mathsf{T}(\mathbf{L})\mathbf{X}$
6	$\mathcal{X}, \mathcal{A}$	$\mathcal{X} \in \mathcal{A}$	1	$d = \mathbf{X}^\mathsf{T}\mathbf{A} = \mathbf{A}^\mathsf{T}\mathbf{X}$
7	$\mathcal{L}, \mathcal{M}$	$\mathcal{L} \equiv \mathcal{M}$	4	$\mathrm{D} = \overline{\Gamma}(\mathbf{L})\Gamma(\mathbf{M})$
8		$\mathcal{L} \cap \mathcal{M} \neq \emptyset$	1	$d = \overline{\mathbf{L}}^\mathsf{T}\mathbf{M} = \overline{\mathbf{M}}^\mathsf{T}\mathbf{L}$
9	$\mathcal{L}, \mathcal{A}$	$\mathcal{L} \in \mathcal{A}$	2	$\mathbf{D} = \Pi^\mathsf{T}(\mathbf{A})\mathbf{L} = \Gamma^\mathsf{T}(\mathbf{L})\mathbf{A}$
10	$\mathcal{A}, \mathcal{B}$	$\mathcal{A} \equiv \mathcal{B}$	3	$\mathbf{D} = \Pi(\mathbf{A})\mathbf{B} = -\Pi(\mathbf{B})\mathbf{A}$

Tabelle: *shows 7 relationships between points, lines and planes useful for 3D grouping, together with the degree of freedom and the essential part of the test statistic.*

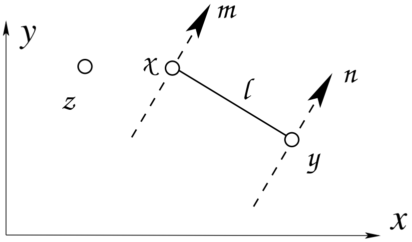
1	2	3	4	5
No.	entities	relation	dof	test
1	χ', $\mathcal{P}$(P), $\mathcal{X}$	$\chi' \equiv \mathcal{P}(\mathcal{X})$	2	$\mathbf{d} = \mathsf{S}(\mathbf{x}')\mathsf{P}\mathbf{X} = \mathbf{0}$
2	l', $\mathcal{P}$(P), $\mathcal{L}$	$l' \equiv \mathcal{P}(\mathcal{L})$	2	$\mathbf{D} = \Gamma(\mathbf{L})\mathsf{P}^\mathsf{T}\mathbf{l}' = \mathbf{0}$
3	$\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, $\mathcal{T}$	coplanar	1	$d = \lvert\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{T}\rvert = 0$
4	$\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$	intersect	1	$d = \lvert\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\rvert = 0$

Tabelle: *shows 4 multi linear relationships together with the degree of freedom and the essential part of the test statistic.*

Grouping



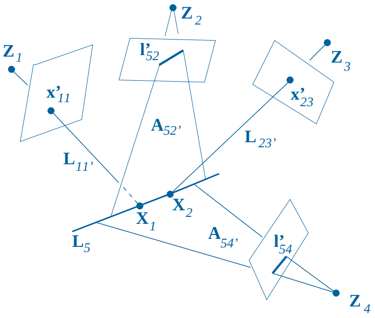
Intermediate step:
Given: edge segment $s(x, y))$, point z
Unknown: Does $z \in s$ hold?



Tests with three lines (l, m, n):

$$\mathbf{z}^\mathsf{T}\mathbf{l} = 0 \qquad \text{sign}\left(\frac{\mathbf{z}^\mathsf{T}\mathbf{m}}{|m_0|}\right) \neq \text{sign}\left(\frac{\mathbf{z}^\mathsf{T}\mathbf{n}}{|n_0|}\right)$$

Combined estimation of a 3D-line



$$
\begin{bmatrix} \overline{\mathbf{L}_{11'}^{\mathsf{T}}} \\ \Pi^{\mathsf{T}}(\mathbf{A}_{52'}^{\mathsf{T}}) \\ \overline{\mathbf{L}_{23'}^{\mathsf{T}}} \\ \Pi^{\mathsf{T}}(\mathbf{A}_{54'}^{\mathsf{T}}) \end{bmatrix} \mathbf{L}_5 = \begin{bmatrix} \mathbf{x'}_{11}^{\mathsf{T}} \mathsf{Q}_1 \\ \Pi^{\mathsf{T}}(\mathsf{P}_2^{\mathsf{T}} \mathbf{l}'_{52}) \\ \mathbf{x'}_{23}^{\mathsf{T}} \mathsf{Q}_3 \\ \Pi^{\mathsf{T}}(\mathsf{P}_4^{\mathsf{T}} \mathbf{l}'_{54}) \end{bmatrix} \mathbf{L}_5 = \mathsf{A}\mathbf{L}_5 = \mathbf{w} = \mathbf{0} \quad \text{SVD}
$$

of A $\rightarrow \widehat{\mathbf{L}_5}$

Result

- Integration of geometry and uncertainty
- Homogeneous representation suited
- Software SUGR (statistically uncertain geometric reasoning) in JAVA available

Use

- Grouping of image and space features
- Reconstruction from images
- Reconstruction from laser range data

Open problems

- Limitations of approach
- Quality of reasoning for ling chains
- Integration of other types of uncertainties (Correspondence, grouping, ...)

How can we determine the covariance matrix between different entities?

Example: Given three 3D-points $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$, and a plane $\mathbf{A}$

1. Determine lines

$$\mathbf{L} = \mathbf{X} \wedge \mathbf{Y} \qquad \mathbf{M} = \mathbf{X} \wedge \mathbf{Z}$$

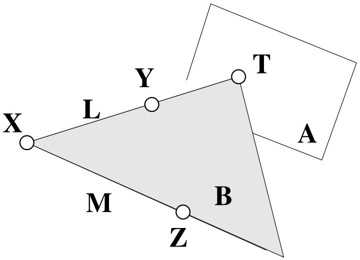
2. Determine fourth point

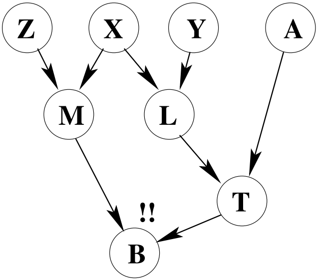
$$\mathbf{T} = \mathbf{L} \cap \mathbf{A}$$

3. Determine plane

$$\mathbf{B} = \mathbf{M} \wedge \mathbf{T}$$

Plane $\mathbf{M} \wedge \mathbf{T}$ should be identical to plane $\mathbf{X} \wedge \mathbf{Y} \wedge \mathbf{Z}$
Line $\mathbf{M}$ and point $\mathbf{T}$ both depend on $\mathbf{X}$: $\Sigma_{MT} \neq 0$.
If covaraince Σ_{MT} is neglected, then $D(\mathbf{M} \wedge \mathbf{T}) \neq D(\mathbf{X} \wedge \mathbf{Y} \wedge \mathbf{Z})$





Construction of plane $\mathbf{B} = \mathbf{M} \wedge \mathbf{T}$ with $\mathbf{M} = \mathbf{X} \wedge \mathbf{Z}$ and $\mathbf{T} = \mathbf{A} \cap (\mathbf{X} \wedge \mathbf{Y})$

General setup:

Given:

– mutually independent vectors $(\underline{\boldsymbol{x}}, \Sigma_{xx})$, $(\underline{\boldsymbol{y}}, \Sigma_{yy})$ and $(\underline{\boldsymbol{z}}, \Sigma_{zz})$

– linear functions

$$\boldsymbol{u} = A\boldsymbol{x} + B\boldsymbol{b}$$
$$\boldsymbol{v} = C\boldsymbol{x} + D\boldsymbol{c}$$

The covariance matrix of $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$ is given by:

$$\Sigma_{uv} = A\Sigma_{xx}C^{\mathsf{T}}$$

Proof:
from

$$\boldsymbol{z} = E\boldsymbol{t}$$

with

$$\boldsymbol{t} = \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \end{bmatrix} \qquad E = \begin{bmatrix} A & B & 0 \\ C & 0 & D \end{bmatrix} \qquad \boldsymbol{z} = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix}$$

we obtain

$$\Sigma_{zz} = E\,\Sigma_{tt}E^{\mathsf{T}}$$

with

$$\Sigma_{zz} = \begin{bmatrix} \Sigma_{uu} & \Sigma_{uv} \\ \Sigma_{vu} & \Sigma_{vv} \end{bmatrix} = \begin{bmatrix} A & B & 0 \\ C & 0 & D \end{bmatrix} \begin{bmatrix} \Sigma_{xx} & 0 & 0 \\ 0 & \Sigma_{yy} & 0 \\ 0 & 0 & \Sigma_{zz} \end{bmatrix} \begin{bmatrix} A^{\mathsf{T}} & C^{\mathsf{T}} \\ B^{\mathsf{T}} & 0 \\ 0^{\mathsf{T}} & D^{\mathsf{T}} \end{bmatrix}$$