Derivation of Linear Kalman Filter – from Gilbert Strang (MIT)

$\boxed{\text{Linear Algebra, Geodesy and GPS}}$

$B_i x_i \approx \ell_i$ obs/cond. eqn, epoch i, weight $= W_o$

$\Phi_{i,i+1} x_i = x_{i+1}$ state transition equation, epoch $i \rightarrow$ epoch $i+1$

$-\Phi_{i,i+1} x_i + x_{i+1} = 0$ pseudo observation, 0, weight $= W_t$

consider 4 epochs:

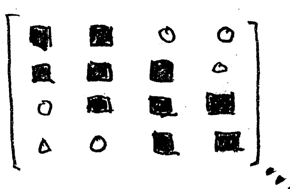
$$\begin{bmatrix} B_1 & & & \\ -\Phi_{12} & I & & \\ & B_2 & & \\ & -\Phi_{23} & I & \\ & & B_3 & \\ & & -\Phi_{34} & I \\ & & & B_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \approx \begin{bmatrix} \ell_1 \\ 0 \\ \ell_2 \\ 0 \\ \ell_3 \\ 0 \\ \ell_4 \end{bmatrix}, \quad W = \begin{bmatrix} W_o & & & & & & \\ & W_t & & & & & \\ & & W_o & & & & \\ & & & W_t & & & \\ & & & & W_o & & \\ & & & & & W_t & \\ & & & & & & W_o \end{bmatrix},$$

form normal equations

$$\begin{bmatrix} B_1^T & -\Phi_{12}^T & & & & & \\ & I & B_2^T & -\Phi_{23}^T & & & \\ & & & I & B_3^T & -\Phi_{34}^T & \\ & & & & & I & B_4^T \end{bmatrix} \begin{bmatrix} W_o & & & & & & \\ & W_t & & & & & \\ & & W_o & & & & \\ & & & W_t & & & \\ & & & & W_o & & \\ & & & & & W_t & \\ & & & & & & W_o \end{bmatrix} \begin{bmatrix} B_1 & & & \\ -\Phi_{12} & I & & \\ & B_2 & & \\ & -\Phi_{23} & I & \\ & & B_3 & \\ & & -\Phi_{34} & I \\ & & & B_4 \end{bmatrix} =$$

$$\begin{bmatrix} B_1^T W_o B_1 + \Phi_{12}^T W_t \Phi_{12} & -\Phi_{12}^T W_t & 0 & 0 \\ -W_t \Phi_{12} & W_t + B_2^T W_o B_2 + \Phi_{23}^T W_t \Phi_{23} & -\Phi_{23}^T W_t & 0 \\ 0 & -W_t \Phi_{23} & W_t + B_3^T W_o B_3 + \Phi_{34}^T W_t \Phi_{34} & -\Phi_{34}^T W_t \\ 0 & 0 & -W_t \Phi_{34} & W_t + B_4^T W_o B_4 \end{bmatrix}$$

$$\begin{bmatrix} N_{11} & N_{12} & 0 & 0 \\ N_{21} & N_{22} & N_{23} & 0 \\ 0 & N_{32} & N_{33} & N_{34} \\ 0 & 0 & N_{43} & N_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix}$$



Block Tri-Diagonal

$N_{11} x_1 + N_{12} x_2 = t_1$

$N_{11} x_1 = t_1 - N_{12} x_2$

$x_1 = N_{11}^{-1}(t_1 - N_{12} x_2)$

$N_{21} x_1 + N_{22} x_2 + N_{23} x_3 = t_2$

$N_{21} N_{11}^{-1}(t_1 - N_{12} x_2) + N_{22} x_2 + N_{23} x_3 = t_2$

$\underbrace{(N_{22} - N_{21} N_{11}^{-1} N_{12})} x_2 + N_{23} x_3 = \underbrace{t_2 - N_{21} N_{11}^{-1} t_1}$

$N_{22}' \, x_2 + N_{23} x_3 = t_2'$

$x_2 = (N_{22}')^{-1}(t_2' - N_{23} x_3)$

$N_{32} x_2 + N_{33} x_3 + N_{34} x_4 = t_3$

$N_{32} (N_{22}')^{-1} (t_2' - N_{23} x_3) + N_{33} x_3 + N_{34} x_4 = t_3$

$\underbrace{(N_{33} - N_{32} (N_{22}')^{-1} N_{23})} x_3 + N_{34} x_4 = \underbrace{t_3 - N_{32} (N_{22}')^{-1} t_2}$

$N_{33}' \, x_3 + N_{34} x_4 = t_3'$

$x_3 = (N_{33}')^{-1} (t_3' - N_{34} x_4)$

$N_{43} x_3 + N_{44} x_4 = t_4$

$N_{43} (N_{33}')^{-1} (t_3' - N_{34} x_4) + N_{44} x_4 = t_4$

$\underbrace{(N_{44} - N_{43} (N_{33}')^{-1} N_{34})} x_4 = t_4 - \underbrace{N_{43} (N_{33}')^{-1} t_3'}$

$N_{44}' \, x_4 = t_4'$

$\boxed{x_4 = (N_{44}')^{-1} t_4'}$

When epoch 5 occurs, eliminate x_4, and solve numerically for x_5, etc.