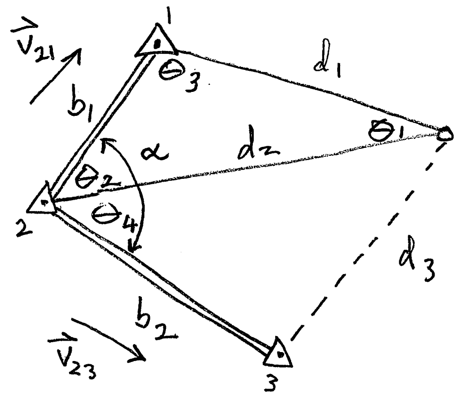
Derive the observations only condition equation for the 3-range problem.



Compute the upper triangle using d_1, d_2, and the fixed side. Then get θ_4 and compute d_3.

$$b_1^2 = d_1^2 + d_2^2 - 2 d_1 d_2 \cos\theta_1$$

$$\frac{-(b_1^2 - d_1^2 - d_2^2)}{2 d_1 d_2} = \cos\theta_1$$

$$\theta_1 = a\cos\left(\frac{-b_1^2 + d_1^2 + d_2^2}{2 d_1 d_2}\right)$$

$$\frac{b_1}{\sin\theta_1} = \frac{d_1}{\sin\theta_2} = \frac{d_2}{\sin\theta_3}, \qquad \frac{\sin\theta_1}{b_1} = \frac{\sin\theta_2}{d_1} = \frac{\sin\theta_3}{d_2}$$

$$\sin\theta_2 = \frac{d_1}{b_1}\sin\theta_1, \qquad \theta_2 = a\sin\left(\frac{d_1}{b_1}\sin\theta_1\right)$$

$$\sin\theta_3 = \frac{d_2}{b_1}\sin\theta_1, \qquad \theta_3 = a\sin\left(\frac{d_2}{b_1}\sin\theta_1\right)$$

$$\boxed{\begin{aligned} \vec{a}\cdot\vec{b} &= |a||b|\cos\theta \\ \vec{V}_{12}\cdot\vec{V}_{23} &= b_1 b_2 \cos\alpha \\ \alpha &= a\cos\left(\frac{\vec{V}_{12}\cdot\vec{V}_{23}}{b_1 b_2}\right) \end{aligned}}$$

$$\theta_4 = \alpha - \theta_2$$

$$d_3^2 = b_2^2 + d_2^2 - 2 b_2 d_2 \cos\theta_4, \qquad F = d_3^2 - b_2^2 - d_2^2 + 2 b_2 d_2 \cos\theta_4$$

Now make substitutions for $\theta_4, \theta_2, \theta_1$:

$$F = d_3^2 - b_2^2 - d_2^2 + 2 b_2 d_2 \cos(\alpha - \theta_2) = 0$$

$$F = d_3^2 - b_2^2 - d_2^2 + 2 b_2 d_2 \cos\left(\alpha - a\sin\left(d_1/b_1 \sin\theta_1\right)\right) = 0$$

$$\boxed{F = d_3^2 - b_2^2 - d_2^2 + 2 b_2 d_2 \cos\left(\alpha - a\sin\left(d_1/b_1 \sin\left(a\cos\left(\frac{-b_1^2 + d_1^2 + d_2^2}{2 d_1 d_2}\right)\right)\right)\right) = 0}$$

$F = u_1 + u_2 \cos(u_3)$

$u_1 = d_3^2 - b_2^2 - d_2^2$

$u_2 = 2 b_2 d_2$

$u_3 = \alpha - a\sin(u_4)$

$u_4 = u_5 \sin(u_6)$

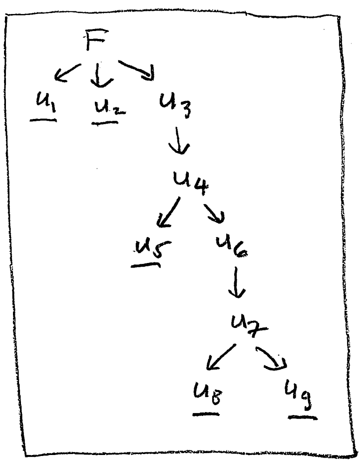
$u_5 = d_1/b_1$

$u_6 = a\cos(u_7)$

$u_7 = u_8/u_9$

$u_8 = -b_1^2 + d_1^2 + d_2^2$

$u_9 = 2 d_1 d_2$



linearization,

$$F = u_1 + u_2 \cos(u_3)$$

$$\frac{\partial F}{\partial x} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x}\cos(u_3) - u_2 \sin(u_3)\frac{\partial u_3}{\partial x}$$

$$\frac{\partial u_3}{\partial x} = \frac{-1}{\sqrt{1-u_4^2}}\frac{\partial u_4}{\partial x}$$

$$\frac{\partial u_4}{\partial x} = \frac{\partial u_5}{\partial x}\sin(u_6) + u_5\cos(u_6)\frac{\partial u_6}{\partial x}$$

$$\frac{\partial u_6}{\partial x} = \frac{-1}{\sqrt{1-u_7^2}}\frac{\partial u_7}{\partial x}$$

$$\frac{\partial u_7}{\partial x} = \frac{u_9\frac{\partial u_8}{\partial x} - u_8\frac{\partial u_9}{\partial x}}{u_9^2}$$

$$\frac{d}{dx}\left(\sin^{-1}u\right) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\cos^{-1}u\right) = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\tan^{-1}u\right) = \frac{1}{1+u^2}\frac{du}{dx}$$

$$\frac{\partial F}{\partial d_3} = 2d_3$$

$\frac{\partial F}{\partial d_1}$:

$$\frac{\partial u_8}{\partial d_1} = 2d_1$$

$$\frac{\partial u_9}{\partial d_1} = 2d_2$$

$$\frac{\partial u_1}{\partial d_1} = 0$$

$$\frac{\partial u_2}{\partial d_1} = 0$$

$$\frac{\partial u_5}{\partial d_1} = \frac{1}{b_1}$$

$\frac{\partial F}{\partial d_2}$:

$$\frac{\partial u_8}{\partial d_2} = 2d_2$$

$$\frac{\partial u_9}{\partial d_2} = 2d_1$$

$$\frac{\partial u_1}{\partial d_2} = -2d_2$$

$$\frac{\partial u_2}{\partial d_2} = 2b_2$$

$$\frac{\partial u_5}{\partial d_2} = 0$$

now start at the bottom, substitute, and work your way up.