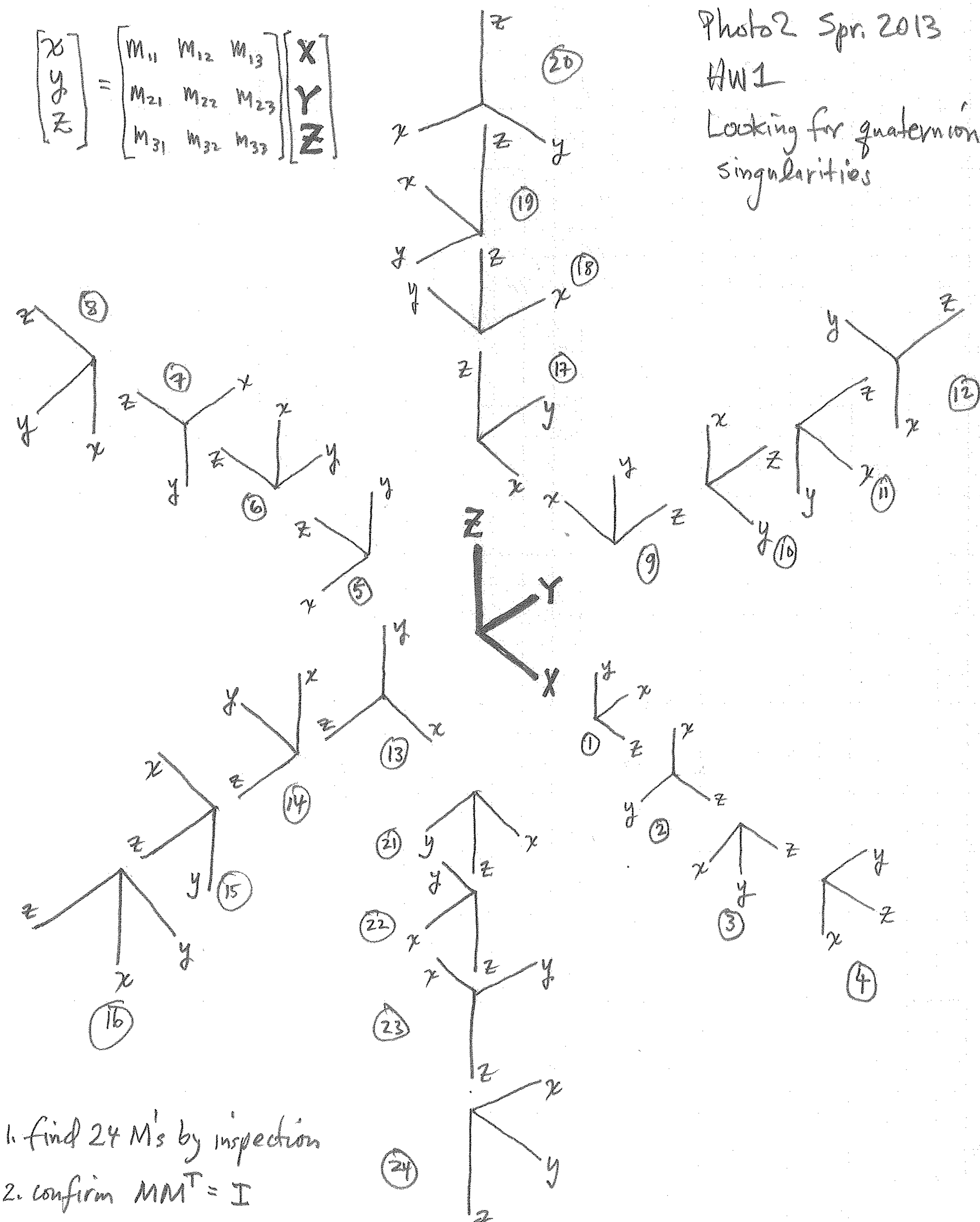
Photo2 Spr. 2013
HW1
Looking for quaternion singularities

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \\ \mathbf{Z} \end{bmatrix}$$



1. find 24 M's by inspection
2. confirm $MM^T = I$
3. extract q_s, q_i, q_j, q_k by algorithm E.2.3
4. do any of 24 cases fail ?

example (1)

$$M = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$4q_s^2 = 1 + m_{11} + m_{22} + m_{33} = 1 + 0 + 0 + 0 = 1$$
$$4q_i^2 = 1 + m_{11} - m_{22} - m_{33} = 1 + 0 - 0 - 0 = 1$$
$$4q_j^2 = 1 - m_{11} + m_{22} - m_{33} = 1 - 0 + 0 - 0 = 1$$
$$4q_k^2 = 1 - m_{11} - m_{22} + m_{33} = 1 - 0 - 0 + 0 = 1$$

find largest magnitude
all the same, choose 1st one

$$4q_s^2 = 1$$
$q_s^2 = 1/4$, $q_s = 0.5$ (either sign ok, choose +) **

Then select the appropriate 3 expressions from (E-43)

$$4\, q_s q_i = m_{32} - m_{23}$$
$$4\, q_s q_j = m_{13} - m_{31}$$
$$4\, q_s q_k = m_{21} - m_{12}$$

$$q_i = \frac{1}{4} \cdot \frac{1}{(0.5)} (0 - 1) = -0.5$$
$$q_j = \frac{1}{4} \cdot \left(\frac{1}{0.5}\right) (0 - 1) = -0.5 \quad \Rightarrow \quad q = \begin{bmatrix} q_s \\ q_i \\ q_j \\ q_k \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \\ -0.5 \\ -0.5 \end{bmatrix}$$
$$q_k = \frac{1}{4} \cdot \left(\frac{1}{0.5}\right) (0 - 1) = -0.5$$

** (q & $-q$ produce the same rotation matrix)

SOLUTION Applying Eqs. E-40 yields

$$q_s = \cos\frac{\theta}{2} = 0.892399$$

$$\sin\frac{\theta}{2} = 0.451247$$

$$\begin{aligned} q_i &= 0.451247(-0.81916073) = -0.369644 \\ q_j &= 0.451247(0.21949345) = 0.0990458 \\ q_k &= 0.451247(-0.52990408) = -0.239118 \end{aligned}$$

$$\boldsymbol{q} = \begin{bmatrix} 0.892399 \\ -0.369644i \\ 0.0990458j \\ -0.239118k \end{bmatrix}$$

E.2.3 Derivation of the Unit Quaternion Corresponding to a Rotation Matrix

By examining the diagonal elements of the quaternion rotation matrix in Eq. E-39 and making use of the relationship $q_s^2 + q_i^2 + q_j^2 + q_k^2 = 1$ for a unit quaternion, we can derive the following relations (Horn, 1987):

$$\begin{aligned} 1 + m_{11} + m_{22} + m_{33} &= 4q_s^2 \\ 1 + m_{11} - m_{22} - m_{33} &= 4q_i^2 \\ 1 - m_{11} + m_{22} - m_{33} &= 4q_j^2 \\ 1 - m_{11} - m_{22} + m_{33} &= 4q_k^2 \end{aligned} \tag{E-42}$$

To maintain numerical precision, we calculate the quaternion component corresponding to the largest term (in absolute value). We then use the relationships between the symmetrically related diagonal elements (e.g., m_{32} and m_{23}) to derive equations for the remaining components:

$$\begin{aligned} m_{32} - m_{23} = 2(q_j q_k + q_s q_i) - 2(q_j q_k - q_s q_i) &= 4q_s q_i \\ m_{13} - m_{31} &= 4q_s q_j \\ m_{21} - m_{12} &= 4q_s q_k \\ m_{21} + m_{12} &= 4q_i q_j \\ m_{32} + m_{23} &= 4q_j q_k \\ m_{13} + m_{31} &= 4q_k q_i \end{aligned} \tag{E-43}$$

EXAMPLE E-4 ***Calculation of Quaternion from Rotation Matrix***

Determine the quaternion corresponding to the following rotation matrix:

$$\boldsymbol{M} = \begin{bmatrix} 0.8660254 & 0.35355339 & 0.35355339 \\ -0.5 & 0.61237244 & 0.61237244 \\ 0.0 & -0.70710678 & 0.70710678 \end{bmatrix}$$