$$\begin{pmatrix} 0 - x_0 \\ -s - y_0 \\ f \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

mm, pixels

$$\frac{0 - x_0}{f} = \frac{u}{w}$$

$$\frac{-s - y_0}{f} = \frac{v}{w}$$

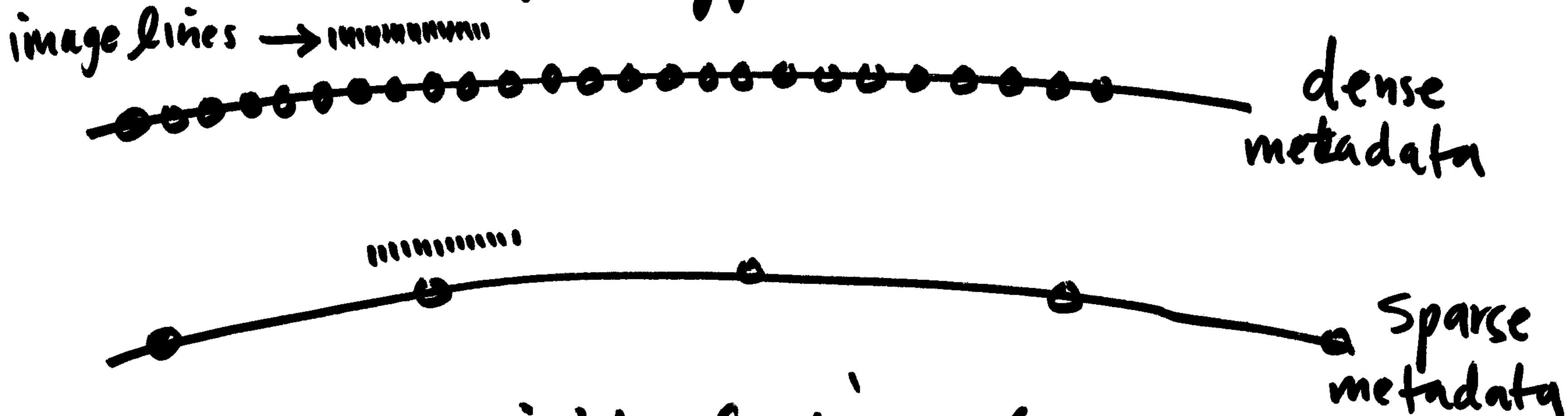
$$0 - x_0 = f\frac{u}{w}$$

$$-s - y_0 = f\frac{v}{w}$$

$\approx \phi$

$$\boxed{\begin{aligned} F_x &= -x_0 - f\frac{u}{w} \\ F_y &= -s - y_0 - f\frac{v}{w} \end{aligned}}$$

metadata, support data



construct initial trajectory

1. interpolate dense sampling (ECF)
2. curve fitting (ECI)
   fit Kepler elements to SV's

$\underline{\Omega}, i, \underline{\omega}, a, e, \dot{\Omega}, \dot{\omega}$

(1)

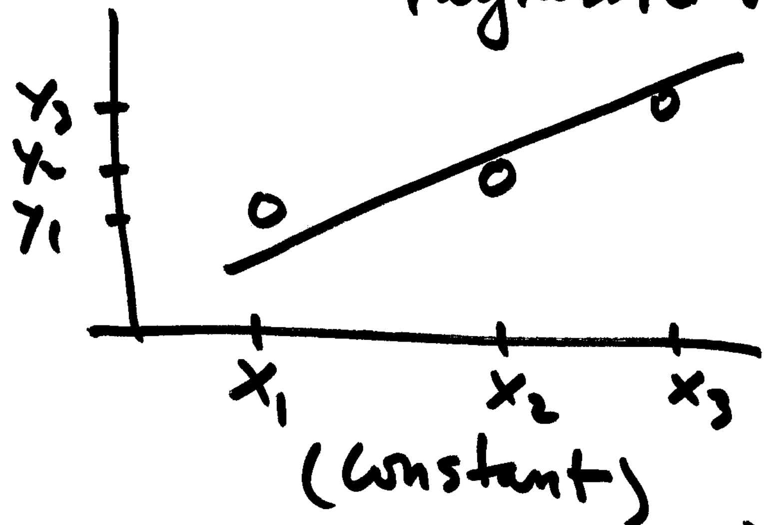
$$\begin{pmatrix} x-x_0 \\ y-y_0 \\ +f \end{pmatrix} = \lambda M_a M_c M \left(\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{ECF} - \left[\begin{pmatrix} X_L \\ Y_L \\ Z_L \end{pmatrix}_{ECF} + \begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix} \right] \right)$$

(2)

$$\begin{pmatrix} x-x_0 \\ y-y_0 \\ +f \end{pmatrix} = \lambda M_a M_c M \left(\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{ECI} - \left[\begin{pmatrix} X_L \\ Y_L \\ Z_L \end{pmatrix}_{ECI} + \begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix} \right] \right)$$

LS: least squares regression

$\dfrac{\text{linear}}{\text{indirect observations}}$ $\stackrel{\text{obs}}{=}$



$X_1 \quad Y_1$

$X_2 \quad Y_2$

$X_3 \quad Y_3$

$Y = mx + b$ m: slope, b: intercept

- overdetermined
- uniquely determined
- underdetermined

	# observations	n
−	min. # to fix model	n_0
	redundancy	r

model : functional part
equations relate obs to parameters

: stochastic part

Condition Equations

write condition equation :

$$V + B\Delta = f$$

vector of corrections or residuals

coefficient matrix

parameter vector

observations

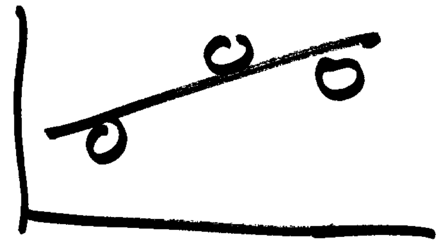
$$y_1 + v_1 = m x_1 + b \qquad v_1 - m x_1 - b = -y_1$$

$$y_2 + v_2 = m x_2 + b \qquad v_2 - m x_2 - b = -y_2$$

$$y_3 + v_3 = m x_3 + b \qquad v_3 - m x_3 - b = -y_3$$

$n = 3$
$n_0 = 2$
$r = 1$

$Y = mx + b$



3 obs.
2 obs $\Rightarrow$ unique solution

$$\left.\begin{aligned} v_1 + y_1 &= m x_1 + b \\ v_2 + y_2 &= m x_2 + b \\ v_3 + y_3 &= m x_3 + b \end{aligned}\right\} \Rightarrow \text{Least Squares}$$

minimize $\sum_i v_i^2$

$w_i = \dfrac{K}{\sigma_i^2} \quad (K = \sigma_0^2)$

minimize $\sum_i w_i v_i^2$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} -x_1 & -1 \\ -x_2 & -1 \\ -x_3 & -1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} -y_1 \\ -y_2 \\ -y_3 \end{bmatrix}$$

$$v + B \Delta = f$$

$$\Delta = (B^T W B)^{-1} B^T W f$$

Least Squares solution

$$v = f - B\Delta$$

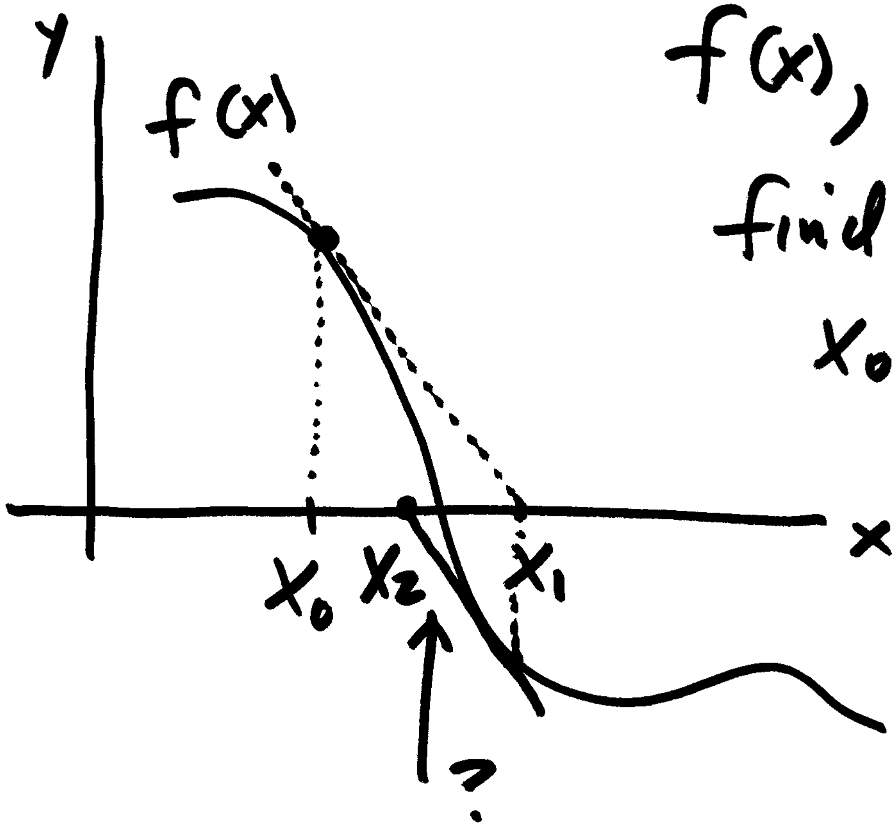
$$Q_{\Delta\Delta} = (B^T W B)^{-1}, \qquad \sigma_0^2 = K$$

$$\Sigma_{\Delta\Delta} = \sigma_0^2 Q_{\Delta\Delta}$$

$$W = \begin{bmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Nonlinear LS

<u>Newton</u> <u>iteration</u>:



$f(x)$, want $x : f(x) = 0$

find root of equation

$x_0, f(x_0), f'(x_0)$

slope = $\frac{\text{rise}}{\text{run}}$

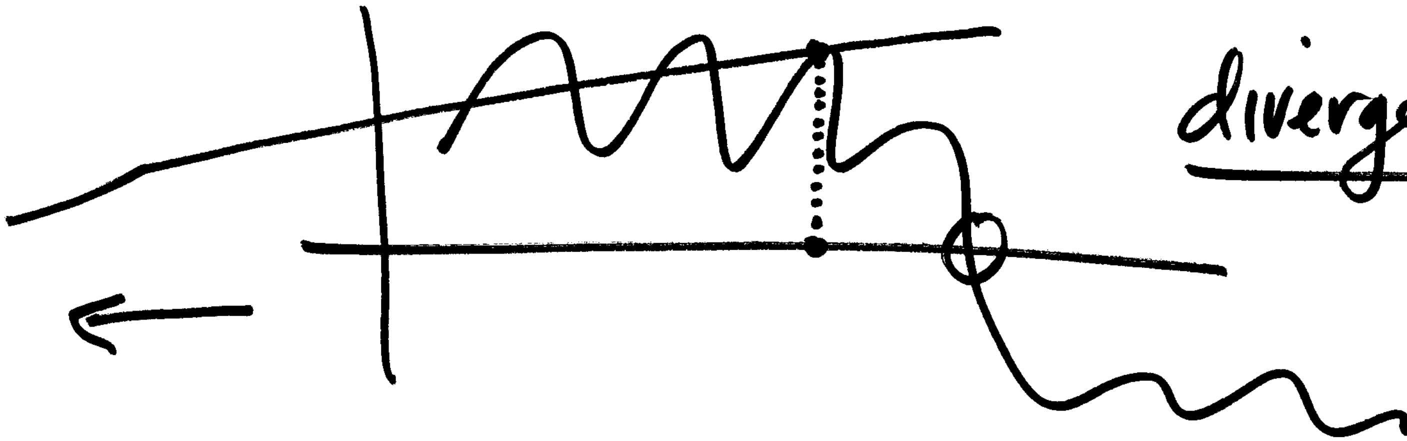
$$\frac{0 - f(x_0)}{x_1 - x_0} = f'(x_0)$$

$$x_1 - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\boxed{x_{i+1} = x_i \underbrace{- \frac{f(x_i)}{f'(x_i)}}_{\text{correction, } \Delta x}}$$

Newton iteration step



divergence

$$\vec{X}_{i+1} = \vec{X}_i - J^{-1} F(x_i)$$

(same principle for n-D vector as for scalar)

$$F_1(x_1, x_2) = 0$$

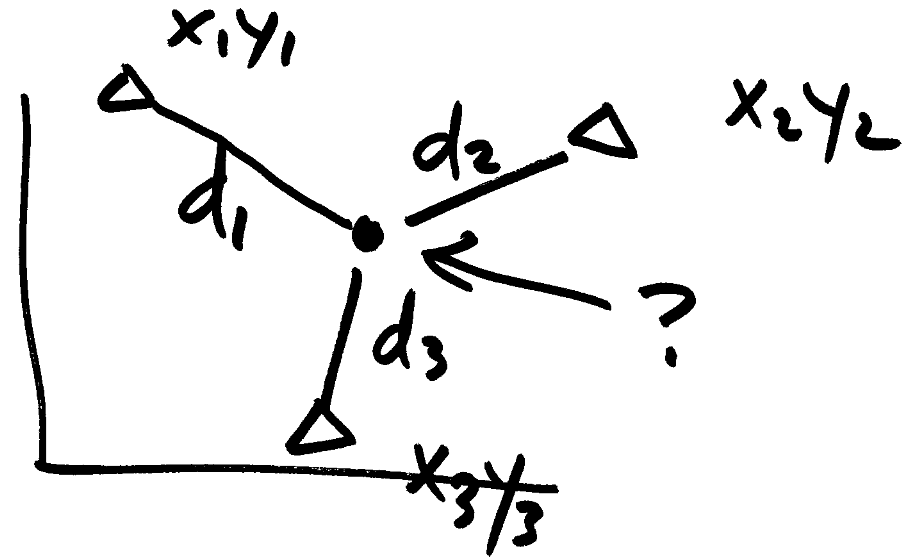
$$F_2(x_1, x_2) = 0$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_{i+1} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_i - \begin{bmatrix} \partial F_1/\partial x_1 & \partial F_1/\partial x_2 \\ \partial F_2/\partial x_1 & \partial F_2/\partial x_2 \end{bmatrix}^{-1} \begin{bmatrix} F_1(x_1, x_2)_i \\ F_2(x_1, x_2)_i \end{bmatrix}$$

multivariate NL LS :
combine linear LS +
multi-var Newton Iteration

$\Rightarrow$ solve multivariate Newton Iteration Problem, but instead of solving uniquely for Δ, we solve a Linear LS problem

Nonlinear LS example



$n = 3$
$n_0 = 2$
$r = 1$

indirect obs. method : choose # unknowns $= n_0$, (x, y)

$$d_1 = \left[(x - x_1)^2 + (y - y_1)^2 \right]^{1/2}$$

$$d_2 = \left[(x - x_2)^2 + (y - y_2)^2 \right]^{1/2}$$

$$d_3 = \left[(x - x_3)^2 + (y - y_3)^2 \right]^{1/2}$$

$$F_1 = d_1 - \left[(x-x_1)^2 + (y-y_1)^2\right]^{1/2} = 0$$

$$F_2 = d_2 - \left[(x-x_2)^2 + (y-y_2)^2\right)^{1/2} = 0$$

$$F_3 = d_3 - \left[(x-x_3)^2 + (y-y_3)^2\right]^{1/2} = 0$$

$$J: \ (B) \quad \begin{bmatrix} \partial F_1/\partial x & \partial F_1/\partial y \\ \partial F_2/\partial x & \partial F_2/\partial y \\ \partial F_3/\partial x & \partial F_2/\partial y \end{bmatrix} \leftarrow \underline{\text{elements of } B}$$

$$J = \begin{bmatrix} \partial F_1/\partial x & \partial F_1/\partial y \\ \partial F_2/\partial x & \partial F_2/\partial y \\ \partial F_3/\partial x & \partial F_3/\partial y \end{bmatrix} = B$$

$$v + \underline{\underline{B}}\,\Delta = f$$

$$f = \begin{bmatrix} -F_1(x_0, y_0) \\ -F_2(x_0, y_0) \\ -F_3(x_0, y_0) \end{bmatrix}$$

→ [note: this slide appeared out of sequence in the lecture — it is part of the 2D range example.]

find partials: 3 ways:

1) analytically

2) numerically (approximation)

3) symbolic processing by software
(results should be same as (1))

partials needed are elements of B (or J).