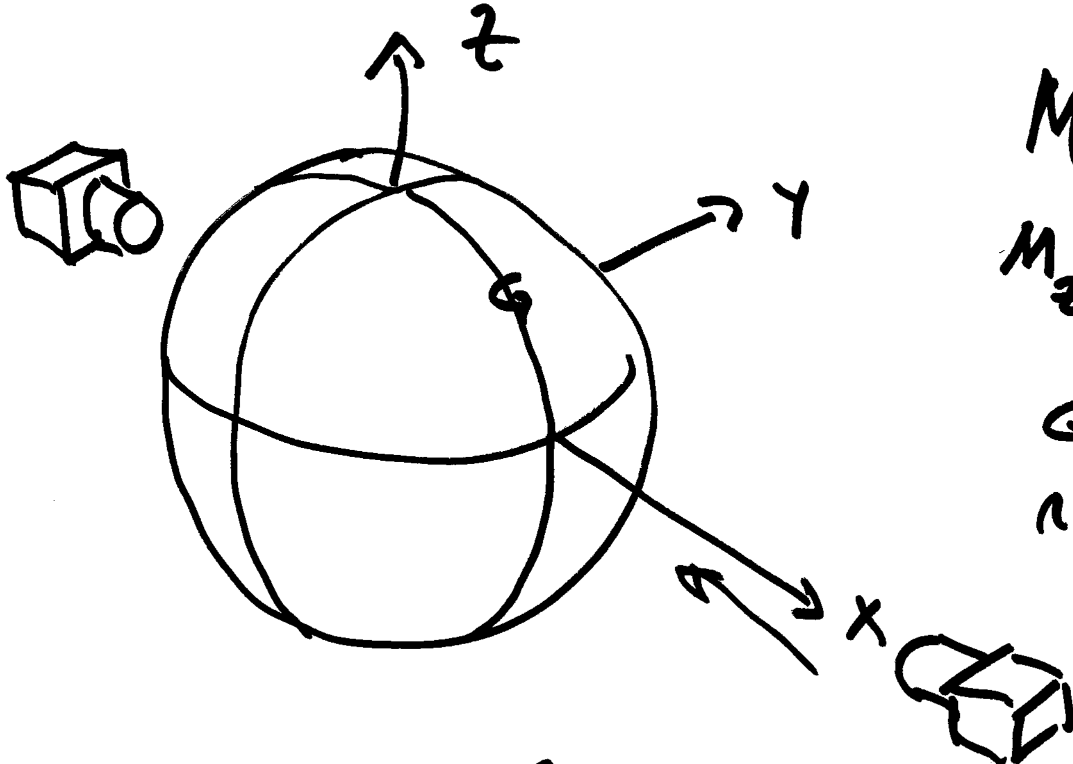


$M_z(k)\, M_y(\phi)\, M_x(\omega)$

$M_z(k)\, M_y(\phi) M_x(\omega)$

↓

Gimbal lock if "middle" rotation is $\pm 90°$

Gimbal Lock

(a) lose a degree of freedom

(b) unable to uniquely extract $\omega\, \phi\, \kappa$ from matrix

euler angles →

axis/angle parameter

quaternions

Derivation of Axis/Angle Parameterization

Euler theorem: any arbitrary rotation represented by (1) axis in space, (2) angle about that axis

(a) columns of rotation matrix: components in "to" system of basis vectors in "from" system

$$\underbrace{Y'}_{\text{to}} = M \underbrace{Y}_{\text{from}}$$

(b) rows of rotation matrix are components in "from" system of basis vectors in "to" system

$$\begin{pmatrix} M_{11} \\ M_{21} \\ M_{31} \end{pmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(c) interpret rotation matrix 2 ways:

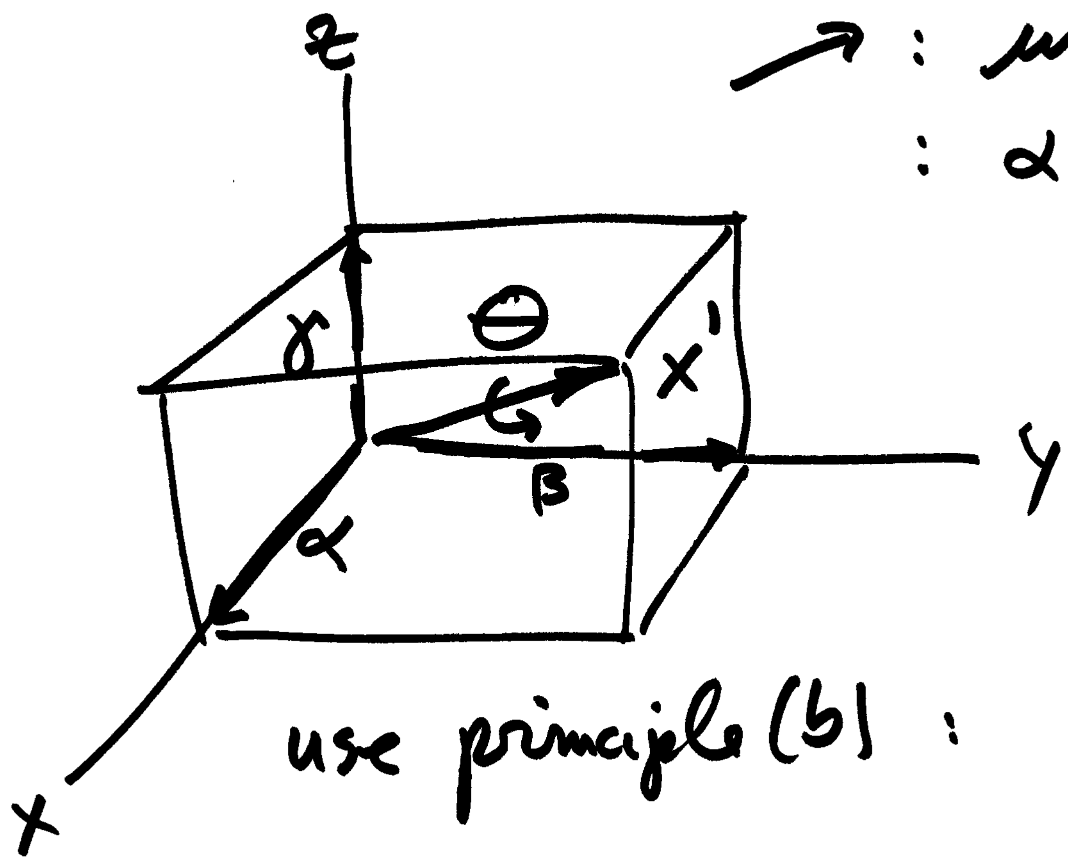
(1) obj. fixed, axes rotate

(2) axes fixed, obj. rotates

(d) rotation matrix : orthogonal

- inner product (dot product) of row or column with itself = 1
- inner product of row with any other row = 0 (also w/cols)

(e) rotation matrix = its cofactor matrix



$\rightarrow$: unit vector

: α, β, γ direction cosines

use principle (b) :

$$\begin{bmatrix} \alpha & \beta & \gamma \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

M_1

$$M_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$M_x^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

by using M_x^T we assume that axes are fixed & object rotated by right-hand rule

now put back to original axi's configuration

$$M_3 : \begin{bmatrix} \alpha & m_{21} & m_{31} \\ \beta & m_{22} & m_{32} \\ \gamma & m_{23} & m_{33} \end{bmatrix}$$

$$M_{\text{axis/angle}} = \begin{bmatrix} \alpha & M_{21} & M_{31} \\ \beta & M_{22} & M_{32} \\ \gamma & M_{23} & M_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \alpha & \beta & \gamma \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$$M_{axis}^{T} \quad M_{,\theta} \quad M_{axis}$$

multiply out symbolically

- inner prod. columns $= 1$
- inner prod. col w/others $= 0$
- element $=$ its cofactor

in the simplification all m_{ij} disappear

(A-68) note: mistake p. 375 text, ok, p. 449

$$M = \begin{bmatrix} \alpha^2(1-\cos\theta)+\cos\theta & \alpha\beta(1-\cos\theta)-\gamma\sin\theta & \alpha\gamma(1-\cos\theta)+\beta\sin\theta \\ \alpha\beta(1-\cos\theta)+\gamma\sin\theta & \beta^2(1-\cos\theta)+\cos\theta & \beta\gamma(1-\cos\theta)-\alpha\sin\theta \\ \alpha\gamma(1-\cos\theta)-\beta\sin\theta & \beta\gamma(1-\cos\theta)+\alpha\sin\theta & \gamma^2(1-\cos\theta)+\cos\theta \end{bmatrix}$$

define axis α, β, γ

rotation θ

<u>Quaternions</u> Appendix E

$$q = \begin{bmatrix} q_s \\ q_i\, i \\ q_j\, j \\ q_k\, k \end{bmatrix} = (q_s, v)$$

$$= q_s + q_i i + q_j j + q_k k$$

multiplication :

$i^2 = j^2 = k^2 = -1$

$ij = k, \; ji = -k$

$jk = i, \; kj = -i$

$ki = j, \; ik = -j$

multiply 2 q's you get ~~anoth~~ another q

dot product $p \cdot q = p_s q_s + p_i q_i + p_j q_j + p_k q_k$

$\|q\|^2 = q \cdot q$, $\|q\| = \sqrt{q \cdot q}$

unit quaternion : $\|q\| = 1$

conjugate :

$$q^* = \begin{bmatrix} q_s \\ -q_i i \\ -q_j j \\ -q_k k \end{bmatrix}$$

$$q q^* = q_s^2 + q_i^2 + q_j^2 + q_k^2 = q \cdot q$$

$$q^{-1} = \frac{q^*}{q \cdot q} = \frac{q^*}{\|q\|^2}$$

if unit q, then $q^{-1} = q^*$

<u>rotation</u>:

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{pmatrix} 0 \\ x \\ y \\ z \end{pmatrix}$$

$$r' = q r q^*$$

$q \to$ matrix? (assume: unit quaternions)

$$M = \begin{bmatrix} q_s^2 + q_i^2 - q_j^2 - q_k^2 & 2(q_j q_i - q_s q_k) & 2(q_i q_k + q_s q_j) \\ 2(q_j q_i + q_s q_k) & q_s^2 - q_i^2 + q_j^2 - q_k^2 & 2(q_j q_k - q_s q_i) \\ 2(q_i q_k - q_s q_j) & 2(q_j q_k + q_s q_i) & q_s^2 - q_i^2 - q_j^2 + q_k^2 \end{bmatrix}$$

$\alpha \beta \gamma \theta \to$ matrix

$\alpha \beta \gamma \theta \to q$

$$q_s = \cos\frac{\theta}{2}$$

$$\begin{pmatrix} q_i \\ q_j \\ q_k \end{pmatrix} = \sin\left(\frac{\theta}{2}\right)\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$