Sequential LS & Derivation of Kalman Filter Equations

10 Nov., 2017

$V + B\Delta = f$, look at formation of normal equations using $\underset{3,2}{B}$ example.

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}, \quad W = \begin{bmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_3 \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}, \quad B^TWB = N, \; B^TWf = t$$

$$B^TW = \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \end{bmatrix} \begin{bmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_3 \end{bmatrix} = \begin{bmatrix} b_{11}w_1 & b_{21}w_2 & b_{31}w_3 \\ b_{12}w_1 & b_{22}w_2 & b_{32}w_3 \end{bmatrix}, \quad \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$N = B^TWB = \begin{bmatrix} \underline{b_{11}w_1b_{11}} + \underline{\underline{b_{21}w_2b_{21}}} + \underline{\underline{\underline{b_{31}w_3b_{31}}}} & \underline{b_{11}w_1b_{12}} + \underline{\underline{b_{21}w_2b_{22}}} + \underline{\underline{\underline{b_{31}w_3b_{32}}}} \\ \underline{b_{12}w_1b_{11}} + \underline{\underline{b_{22}w_2b_{21}}} + \underline{\underline{\underline{b_{32}w_3b_{31}}}} & \underline{b_{12}w_1b_{12}} + \underline{\underline{b_{22}w_2b_{22}}} + \underline{\underline{\underline{b_{32}w_3b_{32}}}} \end{bmatrix}$$

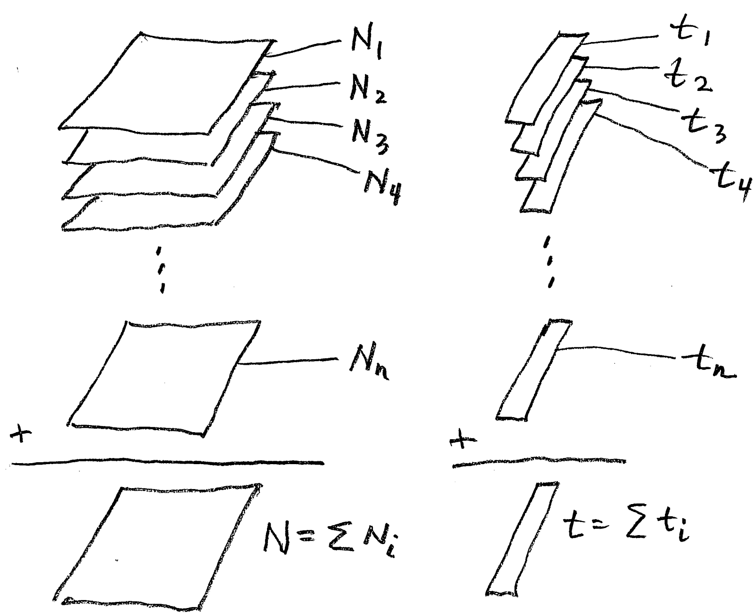
$$t = B^TWf = \begin{bmatrix} \underline{b_{11}w_1f_1} + \underline{\underline{b_{21}w_2f_2}} + \underline{\underline{\underline{b_{31}w_3f_3}}} \\ \underline{b_{12}w_1f_1} + \underline{\underline{b_{22}w_2f_2}} + \underline{\underline{\underline{b_{32}w_3f_3}}} \end{bmatrix},$$

— contribution from first equation

═ contribution from second equation

≡ contribution from third equation

Think of each equation generating a layer to be added in to N and t.



Note: The notation is a bit ambiguous. Is N_i the i^{th} contribution OR the accumulation of N_i up to epoch i? It can mean either! I hope the meaning will be clear from the context.

I will use the notation/symbols from E. Mikhail, but the development sequence from J. Junkins. Equation numbers are from Junkins.

$$N_i = \sum_{j=1}^{i} B_j^T W_j B_j \qquad N_{i-1} = \sum_{j=1}^{i-1} B_j^T W_j B_j \qquad N_i = N_{i-1} + B_i^T W_i B_i$$

$$t_i = \sum_{j=1}^{i} B_j^T W_j f_j \qquad t_{i-1} = \sum_{j=1}^{i-1} B_j^T W_j B_j \qquad t_i = t_{i-1} + B_i^T W_i f_i$$

can update N, t with new contribution

$$N_i^{-1} = \left(N_{i-1} + B_i^T W_i B_i\right)^{-1} \qquad (1.78a)$$

Q: suppose we have Δ_{i-1} and $N_{i-1}^{-1} = Q_{\delta\delta}$ at stage $i-1$, can we compute Δ_i from Δ_{i-1}, and N_i^{-1} from N_{i-1}^{-1} ? A. yes.

We have Δ_{i-1}, N_{i-1}, N_{i-1}^{-1}, and want to obtain Δ_i, N_i, N_i^{-1} for a new observation at epoch i.

$$N\Delta = t, \qquad \Delta = N^{-1} t$$

$$\Delta_i = \left[N_{i-1} + B_i^T W_i B_i\right]^{-1} \left(t_{i-1} + B_i^T W_i f_i\right)$$

$$\Downarrow$$

$$\Delta_i = N_i^{-1} \left(N_{i-1}\Delta_{i-1} + B_i^T W_i B_i\right) \qquad (1.80)$$

$$N_{i-1} = N_i - B_i^T W_i B \qquad (1.81)$$

plug (1.81) into (1.80)

$$\Delta_i = N_i^{-1}\left((N_i - B_i^T W_i B_i)\Delta_{i-1} + B_i^T W_i f_i\right)$$

$$\Delta_i = \underbrace{N_i^{-1} N_i}\Delta_{i-1} - N_i^{-1} B_i^T W_i B_i \Delta_{i-1} + N_i^{-1} B_i^T W_i f_i$$

$$\Delta_i = \Delta_{i-1} + \boxed{N_i^{-1} B_i^T W_i}\,\underbrace{\left(f_i - B_i \Delta_{i-1}\right)} \qquad (1.82)$$

Kalman Gain Matrix K_i

residual vector: i^{th} obs. − prediction from prior [parameter / state vector]

$\Downarrow$

B_i transforms param. → observation
(·) difference of 2 "observations"
K transforms δ obs. → δ param.

} intuitive meaning of terms.

for $K_i = N_i^{-1} B_i^T W_i$, revise the expression for N_i^{-1} using Sherman, Morrison, Woodbury, Schur Formula: $\boxed{(Y+UZV)^{-1} = Y^{-1} - Y^{-1}U(Z^{-1}+VY^{-1}U)^{-1}VY^{-1}}$

$$N_i^{-1} = (N_{i-1} + B_i^T W_i B_i)^{-1} \qquad (1.78a) \qquad \text{apply SMWS ...}$$

$$N_i^{-1} = N_{i-1}^{-1} - N_{i-1}^{-1} B_i^T (Q_i + B_i N_{i-1}^{-1} B_i^T)^{-1} B_i N_{i-1}^{-1} \qquad (1.93)$$

now revise (1.82), subs. new expression for N_i^{-1}

$$\Delta_i = \Delta_{i-1} + \left[N_{i-1}^{-1} - N_{i-1}^{-1} B_i^T (Q_i + B_i N_{i-1}^{-1} B_i^T)^{-1} B_i N_{i-1}^{-1}\right] \underbrace{B_i^T} W_i (f_i - B_i \Delta_{i-1})$$

$$\Delta_i = \Delta_{i-1} + \left[\underbrace{N_{i-1}^{-1} B_i^T} - N_{i-1}^{-1} B_i^T (Q_i + B_i N_{i-1}^{-1} B_i^T)^{-1} B_i N_{i-1}^{-1} B_i^T\right] W_i (f_i - B_i \Delta_{i-1})$$

$$\Delta_i = \Delta_{i-1} + N_{i-1}^{-1} B_i^T \left[\underbrace{I}_{I} - (Q_i + B_i N_{i-1}^{-1} B_i^T)^{-1} B_i N_{i-1}^{-1} B_i^T\right] W_i (f_i - B_i \Delta_{i-1})$$

$$\Delta_i = \Delta_{i-1} + N_{i-1}^{-1} B_i^T \left[\underbrace{(Q_i + B_i N_{i-1}^{-1} B_i^T)^{-1}} (Q_i + B_i N_{i-1}^{-1} B_i^T) - (Q_i + B_i N_{i-1}^{-1} B_i^T)^{-1} B_i N_{i-1}^{-1} B_i^T\right] W_i (f_i - B_i \Delta_{i-1})$$

$$\Delta_i = \Delta_{i-1} + N_{i-1}^{-1} B_i^T (Q_i + B_i N_{i-1}^{-1} B_i^T)^{-1} \left[Q_i + \underbrace{B_i N_{i-1}^{-1} B_i^T - B_i N_{i-1}^{-1} B_i^T}\right] W_i (f_i - B_i \Delta_{i-1})$$

$$\Delta_i = \Delta_{i-1} + \underbrace{\boxed{N_{i-1}^{-1} B_i^T (Q_i + B_i N_{i-1}^{-1} B_i^T)^{-1}}}_{= K_i} (f_i - B_i \Delta_{i-1}) \qquad (1.94)$$

$$K_i = N_{i-1}^{-1} B_i^T (Q_i + B_i N_{i-1}^{-1} B_i^T)^{-1} \qquad \text{revised (1.94)}$$

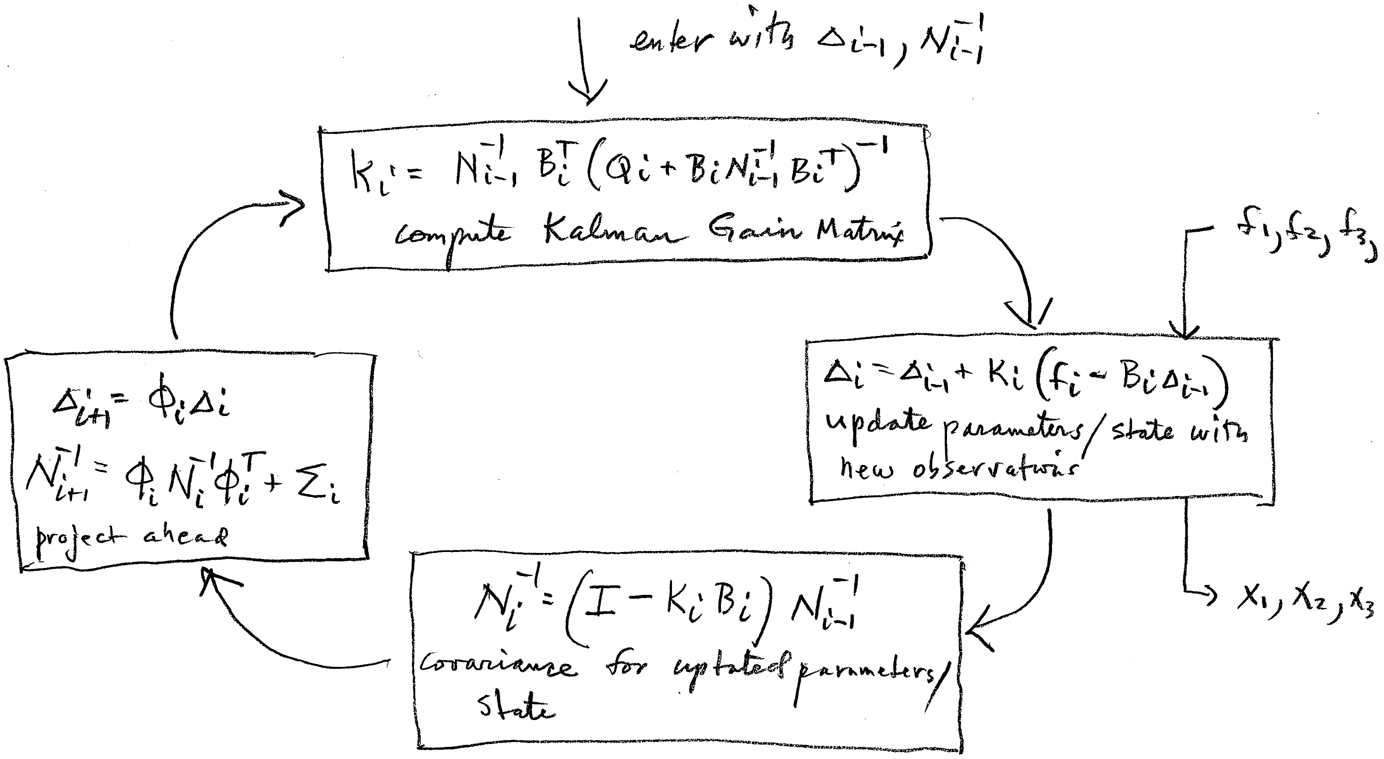
$$\Delta_i = \Delta_{i-1} + K_i (f_i - B_i \Delta_{i-1}) \qquad \text{revised (1.94)}$$

revise (1.93) from top of page above,

$$N_i^{-1} = N_{i-1}^{-1} - K_i B_i N_{i-1}^{-1}$$

$$N_i^{-1} = (I - K_i B_i) N_{i-1}^{-1} \qquad \text{revised (1.93)}$$

all this assumes LINEAR, put into a flowchart like Brown + Hwang,



Brown + Hwang p.219

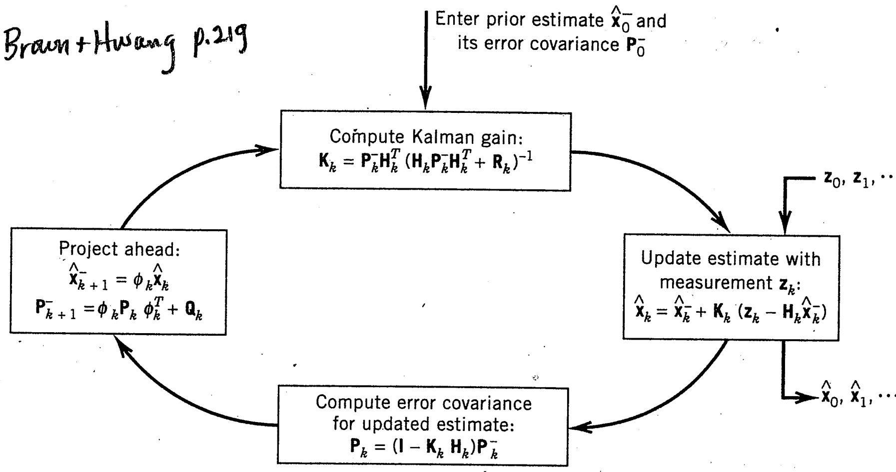


Figure 5.8 Kalman filter loop.

Junkins, J., 1978 An Introduction to Optimal Estimation of Dynamical Systems, Sijthoff and Noordhoff Int'l. Publishers, B.V., Netherlands

Mikhail, E., 1976 Observations and Least Squares, IEP-Dun-Donnelley, New York

Brown, R., and Hwang, P., 1997 Introduction to Random Signals and Applied Kalman Filtering, John Wiley + Sons, New York