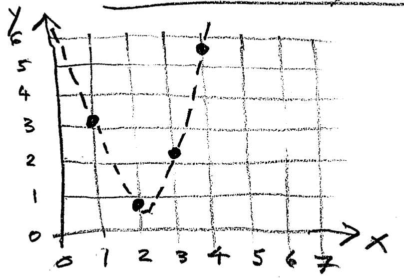
HW1 Solution

10 oct 2018

x	y
1	3.1
2	0.9
3	2.2
4	5.7



$$n = 4$$
$$n_0 = 3$$
$$r = 1$$

$y = \underline{a_0} + \underline{a_1}x + \underline{a_2}x^2 \quad (u = n_0 = 3)$

$\boxed{\text{Ind. Obs}}$: $c = n$, $u = n_0$

choose a_0, a_1, a_2 as parameters

scalar algebra solution (equal weights)

$$y_i + v_i = a_0 + a_1 x_i + a_2 x_i^2$$

$$v_i = a_0 + a_1 x_i + a_2 x_i^2 - y_i$$

plug into objective function:

$$\phi = \sum v_i^2 = (a_0 + a_1 x_1 + a_2 x_1^2 - 3.1)^2 + (a_0 + a_1 x_2 + a_2 x_2^2 - 0.9)^2 + (a_0 + a_1 x_3 + a_2 x_3^2 - 2.2)^2 + (a_0 + a_1 x_4 + a_2 x_4^2 - 5.7)^2$$

$$\frac{\partial \phi}{\partial a_0} = \cancel{2}(a_0 + a_1 + a_2 - 3.1) + \cancel{2}(a_0 + 2a_1 + 4a_2 - 0.9) + \cancel{2}(a_0 + 3a_1 + 9a_2 - 2.2) + \cancel{2}(a_0 + 4a_1 + 16a_2 - 5.7) = 0$$

$$\frac{\partial \phi}{\partial a_1} = \cancel{2}(a_0 + a_1 + a_2 - 3.1) + \cancel{2}(a_0 + 2a_1 + 4a_2 - 0.9)(2) + \cancel{2}(a_0 + 3a_1 + 9a_2 - 2.2)(3) + \cancel{2}(a_0 + 4a_1 + 16a_2 - 5.7)(4) = 0$$

$$\frac{\partial \phi}{\partial a_2} = \cancel{2}(a_0 + a_1 + a_2 - 3.1) + \cancel{2}(a_0 + 2a_1 + 4a_2 - 0.9)(4) + \cancel{2}(a_0 + 3a_1 + 9a_2 - 2.2)(9) + \cancel{2}(a_0 + 4a_1 + 16a_2 - 5.7)(16) = 0$$

$$a_0 + a_0 + a_0 + a_0 + a_1 + 2a_1 + 3a_1 + 4a_1 + a_2 + 4a_2 + 9a_2 + 16a_2 = 3.1 + 0.9 + 2.2 + 5.7$$

$$a_0 + 2a_0 + 3a_0 + 4a_0 + a_1 + 4a_1 + 9a_1 + 16a_1 + a_2 + 8a_2 + 18a_2 + 64a_2 = 3.1 + 1.8 + 6.6 + 22.8$$

$$a_0 + 4a_0 + 9a_0 + 16a_0 + a_1 + 8a_1 + 27a_1 + 64a_1 + a_2 + 16a_2 + 81a_2 + 256a_2 = 3.1 + 3.6 + 19.8 + 91.2$$

$$4a_0 + 10a_1 + 30a_2 = 11.9$$
$$10a_0 + 30a_1 + 100a_2 = 34.3$$
$$30a_0 + 100a_1 + 354a_2 = 117.7$$

$$\begin{bmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 11.9 \\ 34.3 \\ 117.7 \end{bmatrix}$$

, solve in matlab

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 7.8250 \\ -6.2150 \\ 1.4250 \end{bmatrix}$$

$$v_1 = a_0 + a_1 + a_2 - 3.1 = -0.0650$$
$$v_2 = a_0 + 2a_1 + 4a_2 - 0.9 = 0.1950$$
$$v_3 = a_0 + 3a_1 + 9a_2 - 2.2 = -0.1950$$
$$v_4 = a_0 + 4a_1 + 16a_2 - 5.7 = 0.0650$$

$$\hat{y}_1 = y_1 + v_1 = 3.0350$$
$$\hat{y}_2 = y_2 + v_2 = 1.0950$$
$$\hat{y}_3 = y_3 + v_3 = 2.0050$$
$$\hat{y}_4 = y_4 + v_4 = 5.7650$$

3 ways to get cond. eq. (1) elimination

$$\hat{y}_1 = a_0 + a_1 + a_2$$
$$\hat{y}_2 = a_0 + 2a_1 + 4a_2$$
$$\hat{y}_3 = a_0 + 3a_1 + 9a_2$$
$$\hat{y}_4 = a_0 + 4a_1 + 16a_2$$

$$a_0 = \hat{y}_1 - a_1 - a_2$$

$$\hat{y}_2 = \hat{y}_1 - a_1 - a_2 + 2a_1 + 4a_2 = \hat{y}_1 + a_1 + 3a_2$$
$$\hat{y}_3 = \hat{y}_1 - a_1 - a_2 + 3a_1 + 9a_2 = \hat{y}_1 + 2a_1 + 8a_2$$
$$\hat{y}_4 = \hat{y}_1 - a_1 - a_2 + 4a_1 + 16a_2 = \hat{y}_1 + 3a_1 + 15a_2$$

$$a_1 = \hat{y}_2 - \hat{y}_1 - 3a_2$$
$$\hat{y}_3 = \hat{y}_1 + 2(\hat{y}_2 - \hat{y}_1 - 3a_2) + 8a_2 = -\hat{y}_1 + 2\hat{y}_2 + 2a_2$$
$$\hat{y}_4 = \hat{y}_1 + 3(\hat{y}_2 - \hat{y}_1 - 3a_2) + 15a_2 = -2y_1 + 3y_2 + 6a_2$$

$$2a_2 = \hat{y}_3 + \hat{y}_1 - 2\hat{y}_2\,, \qquad a_2 = \tfrac{1}{2}\hat{y}_3 + \tfrac{1}{2}\hat{y}_1 - \hat{y}_2$$
$$\hat{y}_4 = -2y_1 + 3y_2 + 6\left(\tfrac{1}{2}\hat{y}_3 + \tfrac{1}{2}\hat{y}_1 - \hat{y}_2\right)$$
$$\hat{y}_4 = \hat{y}_1 - 3\hat{y}_2 + 3\hat{y}_3$$

$$\boxed{\hat{y}_1 - 3\hat{y}_2 + 3\hat{y}_3 - \hat{y}_4 = 0}$$

$$V_1 - 3V_2 + 3V_3 - V_4 = -(y_1 - 3y_2 + 3y_3 - y_4)$$
$$= -(3.1 - 3(0.9) + 3(2.2) - 5.7)$$
$$= -1.3$$

$$\boxed{V_1 - 3V_2 + 3V_3 - V_4 = -1.3}$$

(2) Lagrange Interpolating Polynomial

$(x_1, y_1)\ (x_2, y_2)\ (x_3, y_3)\ \cdots\ (x_n, y_n)$ n points for an $(n-1)^{th}$ order polynomial fits uniquely

can find any new point by,

$$y = P(x)$$

$$P(x) = \phi_1(x)\, y_1 + \phi_2(x)\, y_2 + \cdots + \phi_n(x)\, y_n \,,$$ where

$$\phi_i(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_1)(x_i-x_2)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

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for our case

$(x_1, y_1)\ (x_2, y_2)\ (x_3\, y_3)$

$$y_4 = P(x_4) = \phi_1(x_4)\, y_1 + \phi_2(x_4)\, y_2 + \phi_3(x_4)\, y_3$$

$$\phi_1(x_4) = \frac{(x_4 - x_2)(x_4 - x_3)}{(x_1 - x_2)(x_1 - x_3)} = \frac{2}{(-1)(-2)} = 1$$

$$\phi_2(x_4) = \frac{(x_4 - x_1)(x_4 - x_3)}{(x_2 - x_1)(x_2 - x_3)} = \frac{(3)}{-1} = -3$$

$$\phi_3(x_4) = \frac{(x_4 - x_1)(x_4 - x_2)}{(x_3 - x_1)(x_3 - x_2)} = \frac{(3)(2)}{(2)} = 3$$

$$y_4 = (1)\cdot y_1 - 3\, y_2 + 3\, y_3$$

$$\hat{y}_1 - 3\hat{y}_2 + 3\hat{y}_3 - \hat{y}_4 = 0$$

$$v_1 - 3v_2 + 3v_3 - v_4 = -(y_1 - 3y_2 + 3y_3 - y_4) = -1.3$$

(3) remember the line equation:

$$\hat{y}_1 - 2\hat{y}_2 + \hat{y}_3 = 0$$

for the second order equation:

$$\hat{y}_1 - 3\hat{y}_2 + 3\hat{y}_3 - \hat{y}_4 = 0$$

look familiar?

coeff. of $(1+x)^n$

$$\binom{n}{k} \quad n \downarrow \quad k \longrightarrow$$

n things taken k at a time

n=0	1					
n=1	1	1				
n=2	1	2	1			
n=3	1	3	3	1		
n=4	1	4	6	4	1	
n=5	1	5	10	10	5	1

⋮

alternate +/- signs to get our coefficients or use

$$(1-x)^n$$

"pascal" triangle

pingala india 5th century BC

al karaji, omar khayyam persia 900–1100

yang hui china 1200

cardano, tartaglia italy 1500's

pascal france 1600's

related topics: polynomials, finite differences, binomial theorem binomial coefficients, combinations, ...

$\boxed{\text{observations only}}$

$$1\hat{y}_1 - 3\hat{y}_2 + 3\hat{y}_3 - 1\hat{y}_4 = 0$$

$n = 4$
$n_0 = 3$
$r = 1$

$\Rightarrow c = r = 1$ condition equation

$$v_1 - 3v_2 + 3v_3 - v_4 = -y_1 + 3y_2 - 3y_3 + y_4$$
$$= -3.1 + 3(0.9) - 3(2.2) + 5.7$$
$$\boxed{v_1 - 3v_2 + 3v_3 - v_4 = -1.3}$$

by substitution: $v_1 = 3v_2 - 3v_3 + v_4 - 1.3$

$$\phi = \sum v_i^2 = (3v_2 - 3v_3 + v_4 - 1.3)^2 + v_2^2 + v_3^2 + v_4^2$$

$$\frac{\partial \phi}{\partial v_2} = \cancel{2}(3v_2 - 3v_3 + v_4 - 1.3)(3) + \cancel{2}v_2 = 0$$

$$\frac{\partial \phi}{\partial v_3} = \cancel{2}(3v_2 - 3v_3 + v_4 - 1.3)(-3) + \cancel{2}v_3 = 0$$

$$\frac{\partial \phi}{\partial v_4} = \cancel{2}(3v_2 - 3v_3 + v_4 - 1.3) + \cancel{2}v_4 = 0$$

$$10v_2 - 9v_3 + 3v_4 = 3.9$$
$$-9v_2 + 10v_3 - 3v_4 = -3.9$$
$$3v_2 - 3v_3 + 2v_4 = 1.3$$

solve by matlab

$$\begin{bmatrix} 10 & -9 & 3 \\ -9 & 10 & -3 \\ 3 & -3 & 2 \end{bmatrix} \begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 3.9 \\ -3.9 \\ 1.3 \end{bmatrix} \qquad \begin{bmatrix} v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0.1950 \\ -0.1950 \\ 0.0650 \end{bmatrix}$$

plug into $v_1 = 3v_2 - 3v_3 + v_4 - 1.3$

$v_1 = -0.0650$

Same result as I/O ✓