Matrix Inversion by partition $A \cdot A^{-1} = I$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & I_n \end{pmatrix}$$

$$A_{11} B_{11} + A_{12} B_{21} = I_m \quad (a)$$

$$A_{11} B_{12} + A_{12} B_{22} = 0 \quad (b)$$

$$A_{21} B_{11} + A_{22} B_{21} = 0 \quad (c)$$

$$A_{21} B_{12} + A_{22} B_{22} = I_n \quad (d)$$

(c) $A_{22} B_{21} = -A_{21} B_{11}$

$$\boxed{B_{21} = -A_{22}^{-1} A_{21} B_{11}}$$

sub into (a)

$$A_{11} B_{11} + A_{12} \left[-A_{22}^{-1} A_{21} B_{11} \right] = I_m$$

$$\left(A_{11} - A_{12} A_{22}^{-1} A_{21} \right) B_{11} = I$$

$$\boxed{B_{11} = \left(A_{11} - A_{12} A_{22}^{-1} A_{21} \right)^{-1}}$$

(d) $A_{22} B_{22} = I - A_{21} B_{12}$

$$\boxed{B_{22} = A_{22}^{-1} - A_{22}^{-1} A_{21} B_{12}}$$

sub into (b)

$$A_{11} B_{12} + A_{12} \left[A_{22}^{-1} - A_{22}^{-1} A_{21} B_{12} \right] = 0$$

(derive expressions for B_{ij} by method #1)

$$(A_{11} - A_{12}A_{22}^{-1}A_{21})B_{12} = -A_{12}A_{22}^{-1}$$

$$B_{12} = -\underbrace{(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}}A_{12}A_{22}^{-1}$$

$$\boxed{B_{12} = -B_{11}A_{12}A_{22}^{-1}}$$

Do again by method #2

start with (b)

$$\boxed{B_{12} = -A_{11}^{-1}A_{12}B_{22}}$$

sub into (d)

$$A_{21}[-A_{11}^{-1}A_{12}B_{22}] + A_{22}B_{22} = I_m$$

$$(A_{22} - A_{21}A_{11}^{-1}A_{12})B_{22} = I$$

$$\boxed{B_{22} = (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}}$$

(a) $A_{11}B_{11} = I - A_{12}B_{21}$

$$\boxed{B_{11} = A_{11}^{-1} - A_{11}^{-1}A_{12}B_{21}}$$

sub into (c)

$$A_{21} A_{11}^{-1} - A_{21} A_{11}^{-1} A_{12} \underline{B_{21}} + A_{22} \underline{B_{21}} = 0$$

$$\left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right) B_{21} = -A_{21} A_{11}^{-1}$$

$$B_{21} = -\underbrace{\left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)^{-1}} A_{21} A_{11}^{-1}$$

$$\boxed{B_{21} = -B_{22} A_{21} A_{11}^{-1}}$$

ok, so by methods #1 & #2 we have 2 expressions for B_{11}. Set them to be equal and re-arrange.

two expressions for B_{11}

$$B_{11} = \underbrace{\left(A_{11} - A_{12} A_{22}^{-1} A_{21}\right)^{-1}} = A_{11}^{-1} - A_{11}^{-1} A_{12} \underbrace{B_{21}}$$

$$= A_{11}^{-1} + A_{11}^{-1} A_{12} \underbrace{B_{22}} A_{21} A_{11}^{-1}$$

$$= \underbrace{A_{11}^{-1} + A_{11}^{-1} A_{12} \left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)^{-1} A_{21} A_{11}^{-1}}$$

$$\boxed{\left(A_{11} - A_{12} A_{22}^{-1} A_{21}\right)^{-1} = A_{11}^{-1} + A_{11}^{-1} A_{12} \left(A_{22} - A_{21} A_{11}^{-1} A_{12}\right)^{-1} A_{21} A_{11}^{-1}}$$

$$(Y - UZV)^{-1} = Y^{-1} + Y^{-1} U \left(Z^{-1} - V Y^{-1} U\right)^{-1} V Y^{-1}$$

$$(Y + UZV)^{-1} = Y^{-1} - Y^{-1} U \left(Z^{-1} + V Y^{-1} U\right)^{-1} V Y^{-1}$$

ok– we have just derived the
S, M, W, S Matrix Inversion Lemma !

$$\boxed{(Y \mp UZV)^{-1} = Y^{-1} \pm Y^{-1}U(Z^{-1} \mp VY^{-1}U)^{-1}VY^{-1}}$$

Sherman Morrison Woodbury Schurr
Matrix Inversion Lemma

now apply this result to sequential LS

$\rightarrow N_i = N_{i-1} + B_i^T W_i B_i$ (notation change)

$t_i = t_{i-1} + B_i^T W_i f_i$

$$N_i = N_{i-1} + B_i^T W_i B_i$$

$$N_i^{-1} = ? = (N_{i-1} + B_i^T W_i B_i)^{-1}$$

$$N_i^{-1} = N_{i-1}^{-1} - N_{i-1}^{-1} B_i^T \left(Q_i + B_i N_{i-1}^{-1} B_i^T\right)^{-1} B_i N_{i-1}^{-1}$$

$$t_i = t_{i-1} + B_i^T W_i f_i$$

$$\Delta_i = \underbrace{N_i^{-1}}\,\underbrace{t_i}$$

$$\Delta_i = \left[N_{i-1}^{-1} - N_{i-1}^{-1} B_i^T \left(Q_i + B_i N_{i-1}^{-1} B_i^T\right)^{-1} B_i N_{i-1}^{-1}\right] \left(t_{i-1} + B^T W_i f_i\right)$$

multiply out & get 4 terms

$+ \quad (A+B)^{-1} = A^{-1}(A^{-1}+B^{-1})^{-1}B^{-1}$

⋮

⋮

$$\Delta_i = \Delta_{i-1} + N_{i-1}^{-1} B_i^T \left(Q_i + B_i N_{i-1}^{-1} B_i^T\right)^{-1} \left(f_i - B_i \Delta_{i-1}\right)$$

update LS when adding observations
also need to add parameters
add dynamic model

Kalman Filter

$$B_i x_i = f_i \quad (W_0)$$

$$x_{i+1} = \Phi_i x_i \ , \quad -\Phi_i x_i + x_{i+1} = 0 \quad (W_t)$$

$$\begin{bmatrix} B_1 & & & \\ -\Phi_1 & I & & \\ \hdashline & B_2 & & \\ & -\Phi_2 & I & \\ \hdashline & & B_3 & \\ & & -\Phi_3 & I \\ \hdashline & & & B_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ 0 \\ \hdashline f_2 \\ 0 \\ \hdashline f_3 \\ 0 \\ \hdashline f_4 \end{bmatrix}$$

$Bx \approx f$

$v + Bx = f$

$\to W = \begin{bmatrix} w_0 & & & \\ & w_t & & \\ & & w_0 & \\ & & & w_t \ddots \end{bmatrix}$

(Gilbert Strang)

form N.E. symbollically

$B^T W B$

$$\left[\begin{array}{c|c|c}
B_1^T W_0 B_1 + \Phi_1 W_t \Phi_1 & -\Phi_1 W_t & 0 \\
\hline
-W_t \Phi_1 & W_t + B_2^T W_0 B_2 + \Phi_2^T W_t \Phi_2 & -\Phi_2 W_t \\
\hline
0 & -W_t \Phi_2 & \ddots \\
\hline
0 & &
\end{array}\right.$$

solve very efficiently by forward elimination

Block Tridiagonal