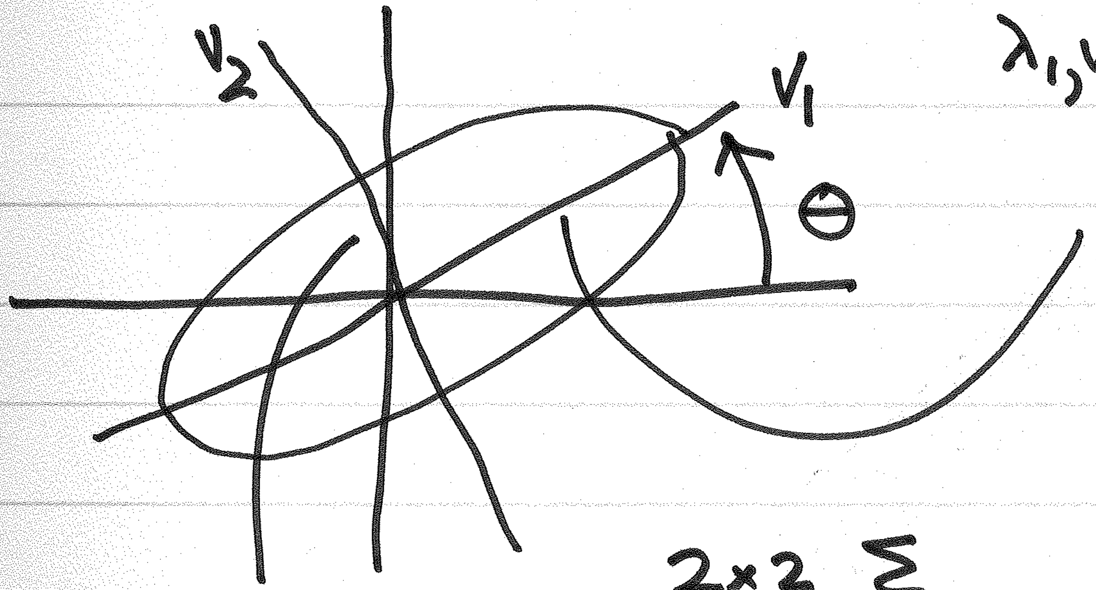
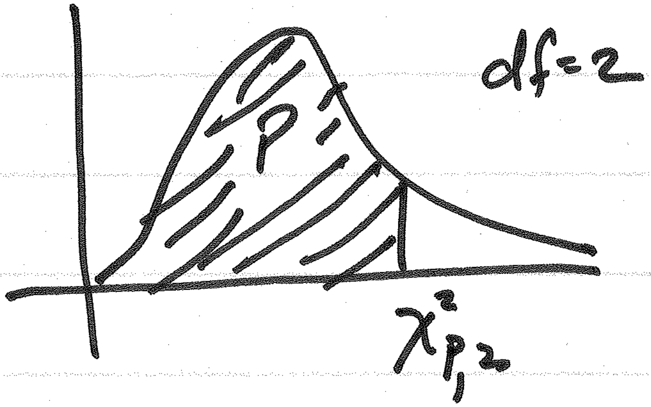


$\lambda_1, v_1 : \lambda_1 > \lambda_2$

$\sqrt{\lambda_1 \chi^2_{p,2}}$

$\sqrt{\lambda} \cdot SF$??



2×2 Σ

$\sqrt{\lambda \chi^2_{p,2}}$ back to characteristic equation $Av = \lambda v$

$\vdots$

→ $\det(A - \lambda I) = 0$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{12} & a_{22} - \lambda \end{vmatrix} = 0 \;, \quad (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}^2 = 0$$

$$a_{11} a_{22} + \lambda^2 - a_{11}\lambda - a_{22}\lambda - a_{12}^2 = 0$$

$$\underset{(A)}{1}\lambda^2 - \underbrace{(a_{11} + a_{22})}_{B}\lambda + \underbrace{a_{11} a_{22} - a_{12}^2}_{C} = 0$$

$$\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}, \quad \lambda = \frac{a_{11} + a_{22}}{2} \pm \frac{\sqrt{(a_{11}+a_{22})^2 - 4(a_{11}a_{22} - a_{12}^2)}}{2}$$

$$\frac{a_{11} + a_{22}}{2} \pm \frac{\sqrt{\underset{\cdots\cdots}{a_{11}^2 + a_{22}^2} + \underline{2a_{11}a_{22} - 4a_{11}a_{22}} + 4a_{12}^2}}{2} \Bigg\} \; -2a_{11}a_{22}$$

$$\frac{a_{11} + a_{22}}{2} \pm \sqrt{\frac{(a_{11} - a_{22})^2 + 4a_{12}^2}{4}} \qquad A : \Sigma : \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix}$$

$$\lambda = \frac{\sigma_x^2 + \sigma_y^2}{2} \pm \sqrt{\frac{(\sigma_x^2 - \sigma_y^2)^2}{4} + \sigma_{xy}^2}$$

λ_1, λ_2

Solve for Θ $\quad AV = V\Lambda, \; A = V\Lambda V^T,$

$A = R^T D R = \Sigma, \; D = R \Sigma R^T$

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix} = \begin{pmatrix} \lambda_1 & \boxed{0} \\ 0 & \lambda_2 \end{pmatrix}$$

$$\begin{bmatrix} c\sigma_x^2 + s\sigma_{xy} & c\sigma_{xy} + s\sigma_y^2 \end{bmatrix} \begin{bmatrix} -s \\ c \end{bmatrix} = 0$$

$$-sc\sigma_x^2 - s^2\sigma_{xy} + c^2\sigma_{xy} + sc\sigma_y^2 = 0$$

$$\sigma_{xy}(c^2 - s^2) + sc(\sigma_y^2 - \sigma_x^2) = 0$$

$$\sigma_{xy}(c^2 - s^2) = sc(\sigma_x^2 - \sigma_y^2)$$

$$\frac{2 \cdot \sigma_{xy}}{\sigma_x^2 - \sigma_y^2} = \frac{2 \cdot sc \div c^2}{c^2 - s^2 \div c^2}$$

$$\frac{2\sigma_{xy}}{\sigma_x^2 - \sigma_y^2} = \frac{2\, s/c}{1 - s^2/c^2} =$$

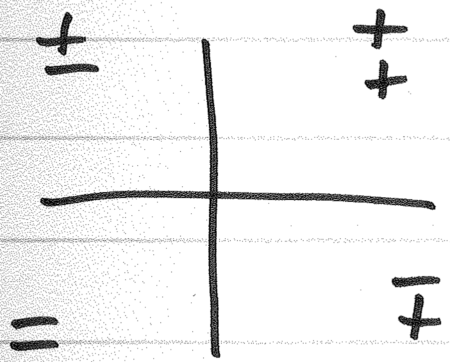
$$= \frac{2\tan\theta}{1 - \tan^2\theta}$$ <u>trig identity</u>

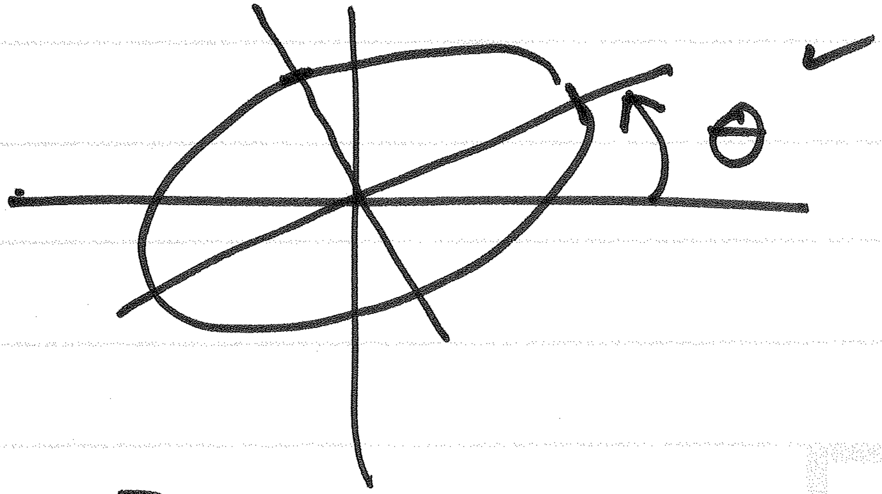
$$= \tan 2\theta$$

$$\tan 2\theta = \frac{2\sigma_{xy}}{\sigma_x^2-\sigma_y^2}, \quad 2\theta = \underline{\tan^{-1}\left(\frac{2\sigma_{xy}}{\sigma_x^2-\sigma_y^2}\right)}$$

get 2θ in correct quadrant based on sign of num & den

$$2\theta = \operatorname{atan2}\left(2\sigma_{xy}, \sigma_x^2-\sigma_y^2\right)$$





$$\text{earlier}\begin{cases}\text{passed Global Test} & z\\ \text{not pass '' ''} & t\end{cases}$$

$$\text{now}\begin{cases}\text{pass} & \chi^2\\ \text{not pass} & F\end{cases}$$

$$y = (\vec{x} - \vec{\mu}_x)^T \hat{\Sigma}_{xx}^{-1} (\vec{x} - \vec{\mu}_x) \sim n \cdot F_{n,r}$$

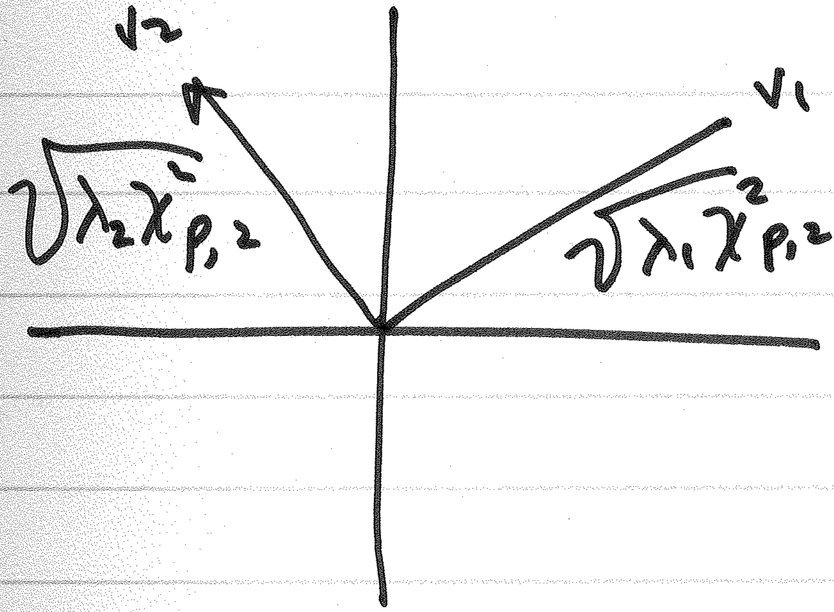
$\Sigma_{xx} = \sigma_0^2 Q_{xx}$

$\hat{\Sigma}_{xx} = \hat{\sigma}_0^2 Q_{xx}$

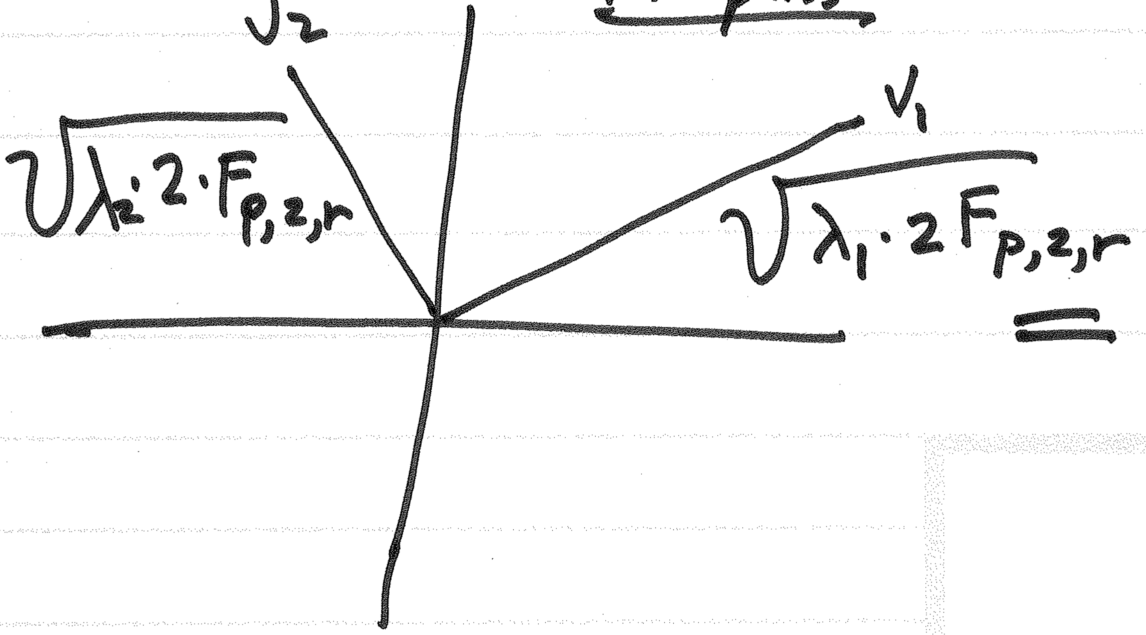
(in our case $n = 2$)

n: dim of vector x

pass

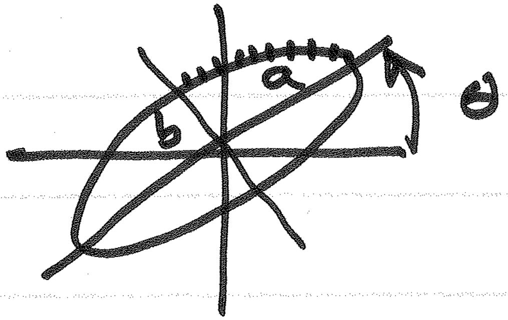


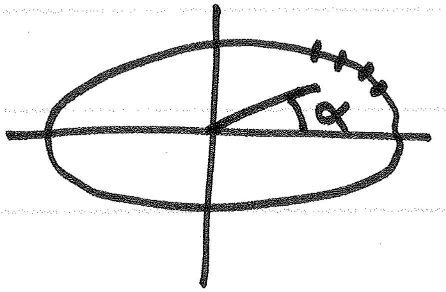
not pass



$$[V, D] = \mathrm{eig}(\Sigma)$$

```
th = Θ
a = ...
b = ...
x0 = a
y0 = b
nseg = 50
dalpha = 2*pi/nseg
for i = 1 : nseg
    alpha = i * dalpha
    x1 = a * cos(alpha)
    y1 = b * sin(alpha)
    px0 = cos(th)*x0 - sin(th)*y0
    Py0 = sin(th)*x0 + cos(th)*y0
    Px1 = cos(th)*x1 - sin(th)*y1
    Px2 = sin(th)*x1 + cos(th)*y1
    plot([px0 px1],[py0 py1],
         '-r');
    if (i==1)
        hold on
    end
    x0 = x1
    y0 = y1
end
```

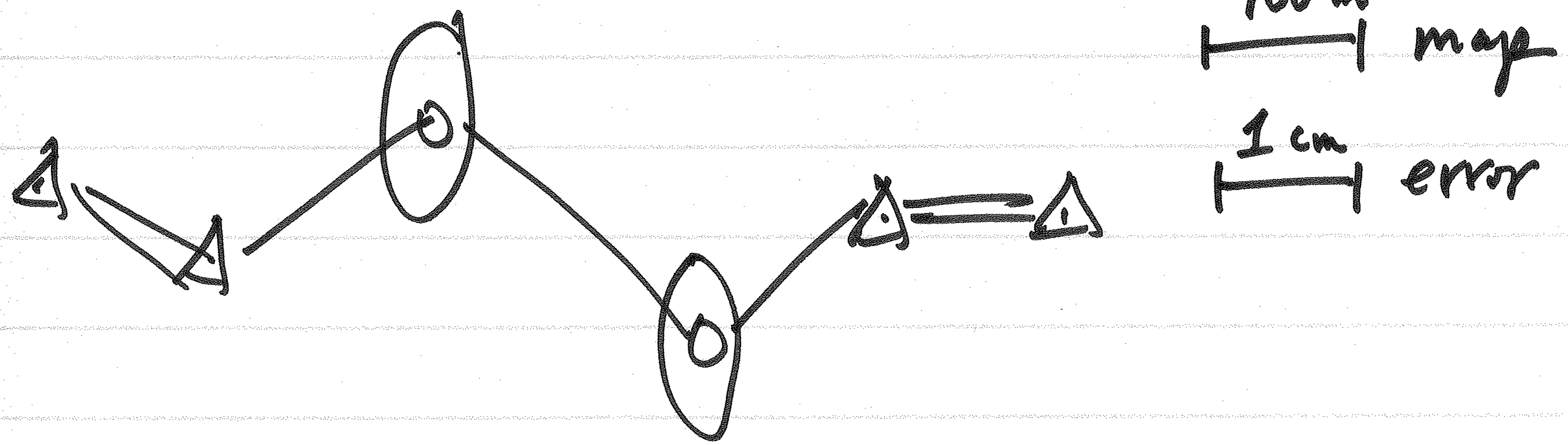




scale $P_{x_0}, P_{y_0}, P_{x_1}, P_{y_1}$

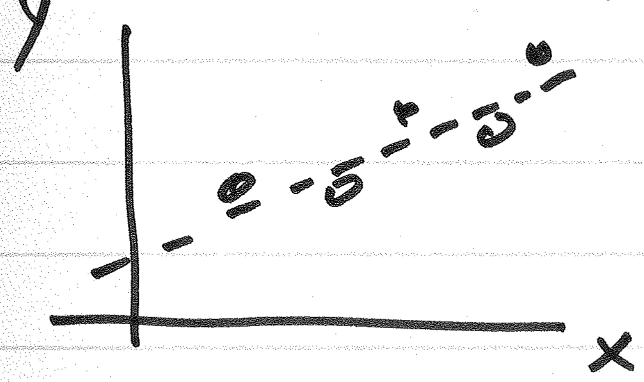
$\underline{\underline{K}}$ factor

so you can see



=

Indirect Obs., Obs. Only, General LS (Mixed Model)



both x & y are observed

5 points, $n = 10$

$n_0 : 4 + 3 = 7$

$r \quad 3$

old way

$n = 5$

$n_0 = 2$

$r = 3$

reconstruct the figure including all observations

$$\underset{\text{obs}}{y} = m\underset{\text{const}}{x} + b$$

I/O

$$\underset{\text{obs}}{y} = \underset{\text{par}}{m}\,\underset{\text{obs}}{x} + \underset{\text{par}}{b} \Rightarrow \underline{\text{NONLINEAR}}$$

GLS

$$F = y - mx - b = 0$$

$$F_1(\ell, x) = 0$$
$$F_2(\ell, x) = 0$$
$$\vdots$$
$$F_c(\ell, x) = 0$$

$$F(\ell, x) \approx \underset{\sim}{F(\ell^0, x^0)} + \underbrace{\frac{\partial F}{\partial \ell}}_{A}\Delta \ell + \underbrace{\frac{\partial F}{\partial x}}_{B}\Delta x = 0$$

$$\ell + v = \ell^0 + \Delta \ell$$
$$\Delta \ell = \ell - \ell^0 + v$$

$$\boxed{\begin{aligned} Av + B\Delta &= -F(\ell^0, x^0) - A(\ell - \ell^0) \\ &= f \end{aligned}}$$

$\boxed{Av + Bo = f}$ GLS / mixed model

maintain 2 vectors l: original

l^0: current observations

#parameters obs. only 0

ind. obs. n_0

GLS $0 < u < n_0$

GLS $c = r + u$

Solve GLS problem
aug. obj. func.

$$\Phi' = v^T W v - 2k^T(Av + B\Delta - f)$$
$$- 2(Av + B\Delta - f)^T k$$

$$\frac{\partial \phi'}{\partial v} = \cancel{2} v^T W - \cancel{2} k^T A = 0$$

$$\frac{\partial \phi'}{\partial \Delta} = -\cancel{2} k^T B = 0$$

$$\frac{\partial \phi'}{\partial k} = -\cancel{2}(Av + B\Delta - f)^T = 0$$

$$Wv - A^T k = 0$$
$$-B^T k = 0$$
$$-(Av + B\Delta - f) = 0$$