Session 16 16-1

note: in the lecture video - please disregard the comments about HW3 #4 and the vector component used for division and elimination of the scale parameter, λ. Those comments are relevant when the equations are re-arranged for intersection, but not for the equations used in problem #4.

Now, resume topic of Error Propagation

1 step & 2 step

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$Y_1 = X_1 + 2X_2$

$Y_2 = 2X_1 + X_2$

$\underline{\underline{\Sigma_{xx} \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}}}$

$Z = Y_1 + 2Y_2$

16-2

1 step

$Z = X_1 + 2X_2 + 2(2X_1 + X_2)$

$\boxed{Z = 5X_1 + 4X_2}$

$Z = \begin{bmatrix} 5 & 4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

$\Sigma_{zz} = A \Sigma_{xx} A^T$

$\sigma_z^2 = \begin{bmatrix} 5 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

$\sigma_z^2 = 204, \quad \sigma_z = \sqrt{204}$

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$y_1 = x_1 + 2x_2$

$y_2 = 2x_1 + x_2$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Sigma_{xx} = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$$

$$\Sigma_{yy} = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}}_{\Sigma_{xx}} \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A^T}$$

$$\underline{\underline{\begin{bmatrix} 24 & 21 \\ 21 & 24 \end{bmatrix}}} = \Sigma_{yy}$$

$$z = y_1 + 2y_2 = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Sigma_{zz} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 24 & 21 \\ 21 & 24 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 204$$

$\sigma_z^2 = 204$, $\sigma_z = \sqrt{204}$

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16-4

$y = Ax$, Σ_{xx}

$y = Ax + b$

matrix of constants (A)

vector of constants (b)

$\Sigma_{yy} = A \Sigma_{xx} A^T$

$\vec{y} = F(\vec{x})$

$$\vec{y} \approx F(x^0) + J \Delta x$$

$\Sigma_{yy} = J \Sigma_{xx} J^T$

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EP for I/O LS method : $\Sigma_{\Delta\Delta} = ?$ 16-5

$\boxed{y = Ax + b}$

Σ_{xx}

Σ_{yy} ?

$\Delta = (B^T W B)^{-1} B^T W f$

$f = d - l$

$\underbrace{f = d - Il}$

$Q_{yy} = A Q_{xx} A^T$

$W,\ Q = W^{-1}$

$Q = Q_{ll}$

$Q_{ff} = (-I)\, Q\, (-I)^T$

$Q_{ff} = Q$

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$\Delta = \underbrace{(B^T W B)^{-1} B^T W}\ \underbrace{f}_{Q}$ 16-6

$\boxed{Q_{\Delta\Delta} = N^{-1}}$

$Q_{\Delta\Delta} = (B^T W B)^{-1} B^T W \cdot Q \cdot \left[(B^T W B)^{-1} B^T W\right]^T$

$= (B^T W B)^{-1} \underbrace{B^T W \underbrace{Q W}_{I} B}_{I} (B^T W B)^{-1}$

$\boxed{\Sigma_{\Delta\Delta} = \sigma_0^2 Q_{\Delta\Delta}}$

if you choose $\sigma_0^2 = 1$, $\Sigma_{\Delta\Delta} = Q_{\Delta\Delta}$

$Q_{\Delta\Delta} = (B^T W B)^{-1}$

$= N^{-1}$

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$$\Sigma_{\Delta\Delta} = \underline{\underline{\sigma_0}}^2 Q_{\Delta\Delta}$$

Estimation

Error Propagation

some common prob. distr. –

normal

MVN

t

χ^2, F

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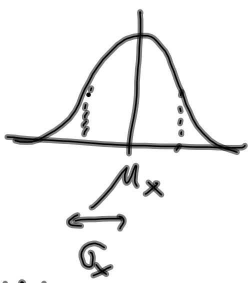
normal distr.

$$\boxed{f(x) = \frac{1}{\sigma_x \sqrt{2\pi}} \cdot \exp\left[\frac{-(x-\mu_x)^2}{2\sigma_x^2}\right]}$$

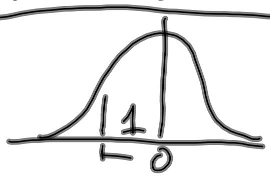


$z = \frac{x-\mu_x}{\sigma_x}$: $\mu_z = 0$, $\sigma_z = 1$

standard normal distr.

$$\boxed{f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right)}$$

probability density function



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MVN multivariate normal distr. 16-9

$$f(\vec{X}) = \frac{1}{(2\pi)^{n/2}\sqrt{|\Sigma|}} \exp\left\{-\frac{1}{2}\cdot(\vec{X}-\vec{\mu}_x)^T \Sigma^{-1} (\vec{X}-\vec{\mu}_x)\right\}$$

$\vec{X}$ $(n,1)$

$$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \vec{\mu}_x \begin{bmatrix} \mu_{x_1} \\ \mu_{x_2} \\ \vdots \\ \mu_{x_n} \end{bmatrix} \quad \Sigma_{xx} = \begin{bmatrix} \sigma_{x_1}^2 & \cdots & & \sigma_{x_1 x_n} \\ & \sigma_{x_2}^2 & & \\ \vdots & & \ddots & \vdots \\ \sigma_{x_n x_1} & \cdots & & \sigma_n^2 \end{bmatrix}$$

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$n = 2$ 16-10

χ^2 chi-squared distribution

$x_1, x_2, \ldots, x_n$ independent, $N(0,1)$

$x_1^2 + x_2^2 + \cdots + x_n^2 = \chi_n^2$

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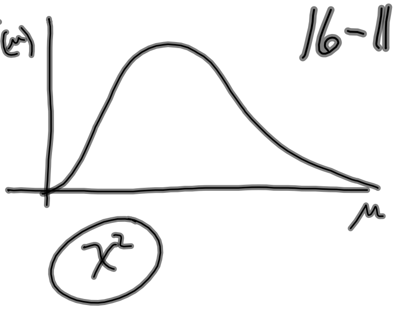
$u : \chi^2_n$

$f(u) = C_n \, u^{(n-2)/2} e^{-u/2}$ for $u > 0$

$C_n = \dfrac{1}{2^{n/2}\Gamma(n/2)}$ Γ: gamma function



$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} \, dt$

$$\left.\begin{array}{l} \Gamma(n+1) = n\,\Gamma(n) \\ \Gamma(1) = 1 \end{array}\right\} \begin{array}{l} \Gamma(n) = (n-1)! \\ \Gamma(n+1) = n! \end{array}$$

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$\bar{X}$, s^2 with sample size n

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$\boxed{\frac{r \cdot \hat{\sigma}_0^2}{\sigma_0^2} \sim \chi^2_r}$$

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