15-1

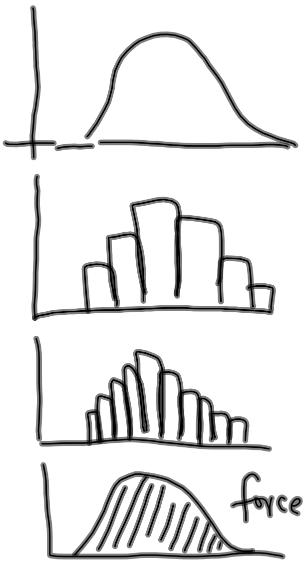
Random Variable : observation

probability density function

normal, gaussian

MVN

t, χ^2, F



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15-2

X, $E(x) = \mu_x$

$$\mu_x : \boxed{E(x) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx}$$

(1st moment about origin)

centroid of density function

Mean

E : expectation

(2nd moment about mean)

$$E\left\{(x-\mu_x)^2\right\} = \int_{-\infty}^{\infty} (x-\mu_x)^2 f(x)\, dx$$

Variance

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variance $E\{(x-\mu_x)^2\} = E\{(x-\mu_x)(x-\mu_x)\}$

σ_x^2, $\sqrt{\text{variance}}$: standard deviation

covariance $E\{(x-\mu_x)(y-\mu_y)\} = \sigma_{xy}$
x, y $= \sigma_{yx}$

$\sigma_x^2 : \sigma_{xy}$

$\frac{\sigma_{xy}}{\sigma_x \sigma_y} = r_{xy}$ correlation coefficient

$-1 \rightarrow +1$

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Expectation operator : Linear

$E(x+y) = E(x) + E(y)$

$E(ax) = aE(x)$

random vector

$$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad E(\vec{X}) = \begin{bmatrix} E(x_1) \\ E(x_2) \\ \vdots \\ E(x_n) \end{bmatrix} = \begin{bmatrix} \mu_{x_1} \\ \mu_{x_2} \\ \vdots \\ \mu_{x_n} \end{bmatrix} = \vec{\mu}_x$$

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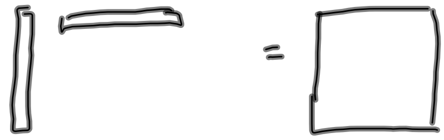
variance/covariance

15-5

$$\Sigma_{xx} = E\{(\vec{X}-\vec{\mu}_x)(\vec{X}-\vec{\mu}_x)^T\}$$



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$$E\left\{ \begin{bmatrix} x_1-\mu_{x_1} \\ x_2-\mu_{x_2} \\ \vdots \\ x_n-\mu_{x_n} \end{bmatrix} \begin{bmatrix} x_1-\mu_{x_1} & x_2-\mu_{x_2} & \cdots & x_n-\mu_{x_n} \end{bmatrix} \right\}$$

$$E\left\{ \begin{bmatrix} (x_1-\mu_{x_1})(x_1-\mu_{x_1}) & (x_1-\mu_{x_1})(x_2-\mu_{x_2}) & \cdots & (x_1-\mu_{x_1})(x_n-\mu_{x_n}) \\ (x_2-\mu_{x_2})(x_1-\mu_{x_1}) & (x_2-\mu_{x_2})(x_2-\mu_{x_2}) & \cdots & \\ \vdots & & & \\ (x_n-\mu_{x_n})(x_1-\mu_{x_1}) & \cdots & & (x_n-\mu_{x_n})(x_n-\mu_{x_n}) \end{bmatrix} \right\}$$

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$$\Sigma_{xx} = \begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1x_2} & \cdots & \sigma_{x_1x_n} \\ \sigma_{x_2x_1} & \sigma_{x_2}^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \sigma_{x_nx_1} & \cdots & \cdots & \sigma_{x_n}^2 \end{bmatrix}$$

variance / covariance matrix

covariance matrix

symmetric matrix

positive semi-definite

$x^T \Sigma_{xx} x \geq 0$ any x

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$\ell, \Sigma_{\ell\ell} \longrightarrow$ LS $\longrightarrow$

estimation	E.P.
x	Σ_{xx}
Δ	$\Sigma_{\Delta\Delta}$
v	Σ_{vv}
$\hat{\ell}$	$\Sigma_{\hat{\ell}\hat{\ell}}$

Σ : precision, dispersion, spread, uncertainty, variability

randomness

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Σ does not describe biases, systematic errors, gross errors, blunders

commonly occuring case:

$$\Sigma_{xx} = \begin{bmatrix} \sigma_{x_1}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{x_2}^2 & & \\ \vdots & & \ddots & \\ 0 & \cdots & & \sigma_{x_n}^2 \end{bmatrix} \text{ diagonal}$$

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$$\Sigma_{xx}^{-1} = \begin{bmatrix} 1/\sigma_{x_1}^2 & 0 & \cdots & 0 \\ 0 & 1/\sigma_{x_2}^2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 1/\sigma_{x_n}^2 \end{bmatrix}$$

$$\sigma_0^2 \Sigma_{xx}^{-1} = \begin{bmatrix} \sigma_0^2/\sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_0^2/\sigma_2^2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \sigma_0^2/\sigma_n^2 \end{bmatrix} =$$

W_{xx}

$W_{xx} = \sigma_0^2 \Sigma_{xx}^{-1}$

$Q_{xx} = \frac{1}{\sigma_0^2} \Sigma_{xx}$

$$\boxed{\Sigma_{xx} = \sigma_0^2 Q_{xx}}$$

Σ_{xx}: cov. matrix

Q_{xx}: scaled version of cov. mx. (cofactor matrix)

Can do E.P. with either Σ or Q

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$\vec{y} = A\vec{x}$ Know Σ_{xx}, what is Σ_{yy} ? 15-11

$\underbrace{E(\vec{y})}_{\mu_y} = E(A\vec{x}) = \underbrace{A\,E(\vec{x})}_{A\mu_x}$ A matrix of constants

$\Sigma_{yy} = E\{(y-\mu_y)(y-\mu_y)^T\}$

($y \uparrow Ax$, $\mu_y \uparrow A\mu_x$)

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$\Sigma_{yy} = E\{(Ax - A\mu_x)(Ax - A\mu_x)^T\}$ Error Propagation 15-12

$(Ax - A\mu_x)^T = (x^TA^T - \mu_x^TA^T)$

$E\{A(x-\mu_x)\underbrace{(x^T-\mu_x^T)}_{(x-\mu_x)^T}A^T\}$

$\Sigma_{yy} = A\,\underbrace{E\{(x-\mu_x)(x-\mu_x)^T\}}_{\Sigma_{xx}}\,A^T$

$$\boxed{\Sigma_{yy} = A\,\Sigma_{xx}\,A^T}$$

$y = Ax$
Σ_{xx}

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examples $\vec{y} = A\vec{x}$ 15-13

$y = a_1 x_1 + a_2 x_2$

assume $\Sigma_{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{bmatrix} \sigma_{x_1}^2 & 0 \\ 0 & \sigma_{x_2}^2 \end{bmatrix}$ ⇐

$y = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\sigma_y^2 = A \Sigma_{xx} A^T = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} \sigma_{x_1}^2 & 0 \\ 0 & \sigma_{x_2}^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$$\boxed{\sigma_y^2 = a_1^2 \sigma_{x_1}^2 + a_2^2 \sigma_{x_2}^2} \checkmark$$

special case

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$y = a_1 x_1 + a_2 x_2$, $\Sigma_{xx} = \begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1 x_2} \\ \sigma_{x_2 x_1} & \sigma_{x_2}^2 \end{bmatrix}$ 15.14

$\sigma_y^2 = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1 x_2} \\ \sigma_{x_2 x_1} & \sigma_{x_2}^2 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

$$\boxed{\sigma_y^2 = a_1^2 \sigma_{x_1}^2 + 2 a_1 a_2 \sigma_{x_1 x_2} + a_2^2 \sigma_{x_2}^2}$$

general result

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$\Sigma : \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}$ $r_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$, $\underline{\sigma_{12} = r_{12} \cdot \sigma_1 \sigma_2}$ 15-15

$\Sigma : \begin{bmatrix} \sigma_1^2 & r_{12}\sigma_1\sigma_2 \\ r_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$

$y_1 = x_1 + 2x_2$

$y_2 = 2x_1 + x_2$

$z = y_1 + 2y_2$

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