10-1

$5x + 3y = 7$ linear in x, y

$x^2 + \sin(y) = 10$ nonlinear in x, y

linearity of variable: at most constant coefficient

for LS problems: unknowns: parameters
residuals (obs.)

in order for a problem to be linear, it must be linear in ALL UNKNOWNS

$$f(x) \approx f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots$$

truncate →

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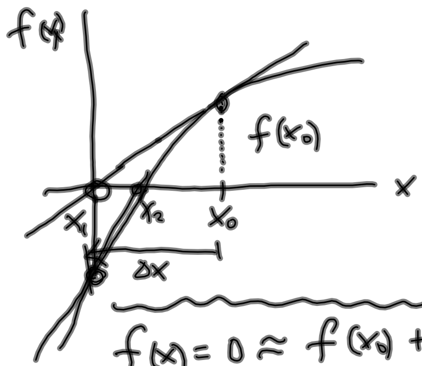
10-2

want to solve $f(x) = 0$ (finding a root)



$$f'(x_0) = \frac{f(x_0)}{-\Delta x}$$

$$\boxed{\Delta x = \frac{-f(x_0)}{f'(x_0)}}$$

Newton Iteration

$$f(x) = 0 \approx f(x_0) + f'(x_0)\,\Delta x \quad (\Delta x = x - x_0)$$

$$-f(x_0) = f'(x_0)\,\Delta x$$

iteration formula

$$\boxed{\Delta x = \frac{-f(x_0)}{f'(x_0)}}$$

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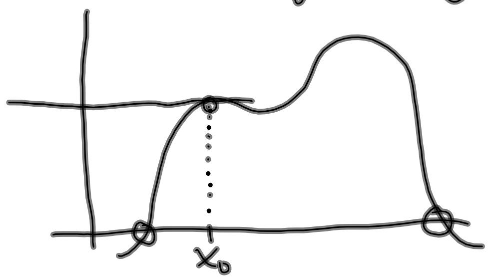
What can go wrong with Newton Iteration? 10-3



1. multiple solutions (+ bad initial approx)
2. solution can <u>diverge</u>
3. incorrect equation, linearize incorrectly

$$f(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

$$f(x_1,x_2,\dots,x_n) = f(x_1^0,x_2^0,\dots,x_n^0) + \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \dots + \frac{\partial f}{\partial x_n}\Delta x_n$$

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$$F_1(x_1,x_2,\dots,x_n) = 0 \approx F_1(x_1^0,x_2^0,\dots,x_n^0) + \frac{\partial F_1}{\partial x_1}\Delta x_1 + \frac{\partial F_1}{\partial x_2}\Delta x_2 + \dots + \frac{\partial F_1}{\partial x_n}\Delta x_n$$

$$F_2(x_1,x_2,\dots,x_n) = 0 \approx F_2(x_1^0,x_2^0,\dots,x_n^0) + \frac{\partial F_2}{\partial x_1}\Delta x_1 + \frac{\partial F_2}{\partial x_2}\Delta x_2 + \dots + \frac{\partial F_2}{\partial x_n}\Delta x_n$$

$$\vdots$$

$$F_n(x_1,x_2,\dots,x_n) = 0 \approx F_n(x_1^0,x_2^0,\dots,x_n^0) + \frac{\partial F_n}{\partial x_1}\Delta x_1 + \frac{\partial F_n}{\partial x_2}\Delta x_2 + \dots + \frac{\partial F_n}{\partial x_n}\Delta x_n$$

$$\begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix} \approx \begin{bmatrix}F_1^0\\F_2^0\\\vdots\\F_n^0\end{bmatrix} + \underbrace{\begin{bmatrix}\partial F_1/\partial x_1 & \partial F_1/\partial x_2 & \cdots & \partial F_1/\partial x_n\\ \partial F_2/\partial x_1 & \partial F_2/\partial x_2 & \cdots & \partial F_2/\partial x_n\\ \vdots & & & \\ \partial F_n/\partial x_1 & \partial F_n/\partial x_2 & \cdots & \partial F_n/\partial x_n\end{bmatrix}}_{\text{Jacobian}} \begin{bmatrix}\Delta x_1\\\Delta x_2\\\vdots\\\Delta x_n\end{bmatrix}$$

$\Delta x = -J^{-1}F(x_0)$ Vector Matrix

$\Delta x = \dfrac{-f(x_0)}{f'(x_0)}$ Scalar

$0 = F(x_0) + J\Delta x$

$-F(x_0) = J\Delta x$

$\boxed{\Delta x = -J^{-1}F(x_0)}$, x_0, $x_1 = x_0 + \Delta x$

$x_{i+1} = x_i + \Delta x$

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10-5

$x^2 + y = 2$ $\quad f_1(x,y) = x^2+y-2 = 0$

$x - 3y^2 = -2$ $\quad f_2(x,y) = x - 3y^2 + 2 = 0$

(1,1)

$\begin{pmatrix} x^0 \\ y^0 \end{pmatrix} : \begin{pmatrix} 1.05 \\ 1.05 \end{pmatrix}$

evaluate $J(x_0,y_0)$ $F(x_0,y_0)$

$$\underbrace{J = \begin{bmatrix} 2x & 1 \\ 1 & -6y \end{bmatrix} \quad F\begin{bmatrix} x^2+y-2 \\ x-3y^2+2 \end{bmatrix}}_{\text{evaluated @ } x^0, y^0}$$

$$J = \begin{bmatrix} 2.10 & 1 \\ 1 & -6.3 \end{bmatrix}, F = \begin{bmatrix} .1525 \\ -.2575 \end{bmatrix}$$

$$\Delta x = -J^{-1} F(x_0) = -\begin{bmatrix} 2.10 & 1 \\ 1 & -6.3 \end{bmatrix}^{-1} \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} -.0494 \\ -.0487 \end{bmatrix}$$

$x_1 = x_0 + \Delta x$

$$\begin{bmatrix} 1.0006 \\ 1.0013 \end{bmatrix} = x_1$$

after 2nd iteration

$$\begin{bmatrix} 1.0000 \\ 1.0000 \end{bmatrix}$$

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<u>conclusion</u> Convergence goes quickly

Converges quadratically

$1, 10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}$

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Big Picture

for NL problems

analyse problem n, n_0, r

modeling, choose x_0

$r=0$	$r>0$
unique solution	LS problem choose I/O O/O
linearize $J, F(x)$	linearize B, f, A, W
solve uniquely $\Delta = -J^{-1}F$	$\delta = (B^T W B)^{-1} B^T W f$ solve LS prob.

$$X_{i+1} = X_i + \Delta$$

repeat until Δ small

newton iteration loop

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Indirect Obs.

10-8

$$\hat{l} = G(x)$$

$$F(\hat{l}, x) = \hat{l} - G(x) = 0$$

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