$$M_\kappa M_\phi M_\omega =$$

$$\begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix} =$$

$$\begin{bmatrix} \cos\phi\cos\kappa & \cos\omega\sin\kappa + \sin\omega\sin\phi\cos\kappa & \cdots \\ -\cos\phi\sin\kappa & \cos\omega\cos\kappa - \sin\omega\sin\phi\sin\kappa & \cdots \\ \sin\phi & -\sin\omega\cos\phi & \cdots \\ \cdots & \cdots & \sin\omega\sin\kappa - \cos\omega\sin\phi\cos\kappa \\ \cdots & \cdots & \sin\omega\cos\kappa + \cos\omega\sin\phi\sin\kappa \\ \cdots & \cdots & \cos\omega\cos\phi \end{bmatrix}$$

[Note correction from video version !]
(m_{32})

GLS : derivation $\quad \phi = V^T W V \to \min.$

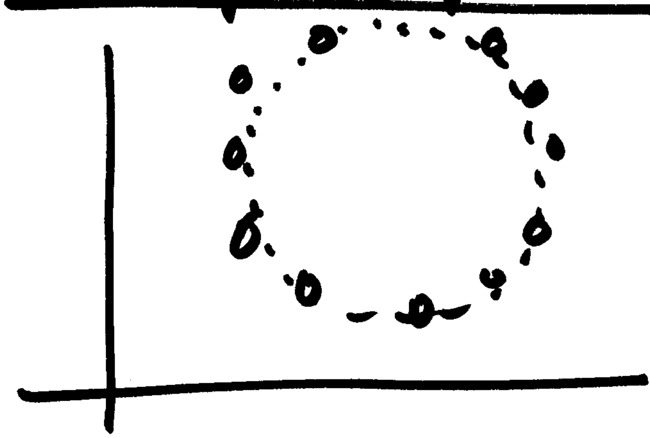
Full Normal Equation

$$\begin{array}{c} (n) \\ (c) \\ (u) \end{array} \begin{bmatrix} -W & A^T & 0 \\ A & 0 & B \\ 0 & B^T & 0 \end{bmatrix} \begin{bmatrix} V \\ K \\ \Delta \end{bmatrix} = \begin{bmatrix} 0 \\ f \\ 0 \end{bmatrix}$$

$\frac{\partial F}{\partial \ell} = A$, $\frac{\partial F}{\partial X} = B$, W, f, $\searrow$ linear eqn

$\underline{n+c+u}$

<u>Examples of GLS</u>



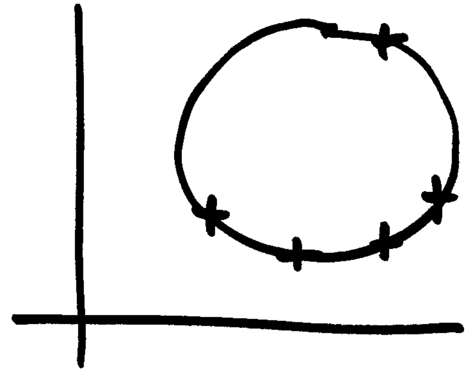
x, y observed

number of points 11, $n = 2 \times 11 = 22$

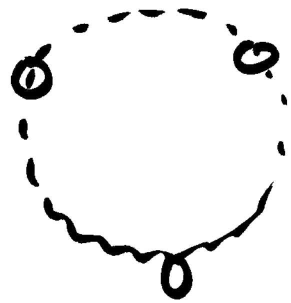
n_0 : $3 + 11 = 14$

$(X_c Y_c R)$

$r = 8$



3 points :



$n = 6$

$n_0 = 3 + 3 = 6$

$r = 0$

I/O always had $\mu = n_0$

GLS μ = anything, $\mu = 3$

$\mu = 3$, $\quad X_c\ Y_c\ R$

$C = r + \mu \qquad r = 8,\ \mu = 3, \quad C = 11$

11 also # points

$\Rightarrow$ write 1 eqn / point

$$(X_i - X_c)^2 + (Y_i - Y_c)^2 = R^2$$

$$F = (X_i - X_c)^2 + (Y_i - Y_c)^2 - R^2 = 0$$

alt

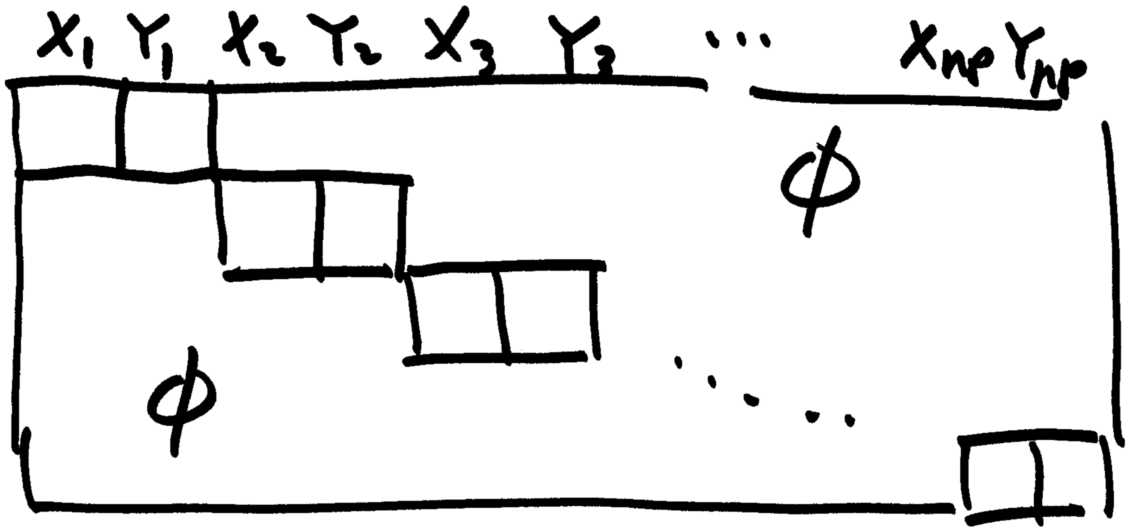
$$F = \left[(X_i - X_c)^2 + (Y_i - Y_c)^2\right]^{1/2} - R = 0 \quad \checkmark$$

$$\frac{\partial F}{\partial X_i} = \cancel{\tfrac{1}{2}}[\cdot]^{-1/2} \cdot \cancel{2}(X_i - X_c)$$

$$= \frac{X_i - X_c}{(\cdot)^{1/2}}$$

$$\frac{\partial F}{\partial y_i} = \cancel{\tfrac{1}{2}}\,(\cdot)^{-1/2} \cdot \cancel{2}\,(y_i - y_c) = \frac{y_i - y_c}{(\cdot)^{1/2}}$$

A
c, n



Block Diagonal

B
c, u
$11, 3$

$$\frac{\partial F}{\partial x_c} = \frac{-(x_i - x_c)}{(\cdot)^{1/2}} \qquad \frac{\partial F}{\partial R} = -1$$

$$\frac{\partial F}{\partial y_c} = \frac{-(y_i - y_c)}{(\cdot)^{1/2}}$$

$$\underset{c,u}{B} \quad \begin{array}{c|c|c|c|} & X_c & Y_c & R \\ \hline \text{point } 1 & -\dfrac{(x_1 - x_c)}{(\cdot)^{1/2}} & -\dfrac{(y_1 - y_c)}{(\cdot)^{1/2}} & -1 \\ \hline \text{point } 2 & -\dfrac{(x_2 - x_c)}{(\cdot)^{1/2}} & -\dfrac{(y_2 - y_c)}{(\cdot)^{1/2}} & -1 \\ \hline & & \vdots & \\ \text{point } np & & & \\ \hline \end{array}$$

$$\underset{c,1}{f}$$

$$f = -F^0 - A(l - l^0)$$

$$f_i = -F_i^0(x_i^0, y_i^0, x_c^0, y_c^0, R^0) - A_i(l - l^0)$$

$$\underline{\underline{f_i}} = \underset{1,1}{-F_i^0(x_i^0, y_i^0, x_c^0, y_c^0, R^0)} - \underset{1,2}{\underset{\text{non zero}}{A_i}} \underset{2,1}{\begin{pmatrix} x_i - x_i^0 \\ y_i - y_i^0 \end{pmatrix}}$$

$u = ?$ Select a value + parameters to make it easy to write cond. eqn.

$u = 7$: $\lambda, \underbrace{\omega, \varphi, \kappa}_{M}, t_x, t_y, t_z$

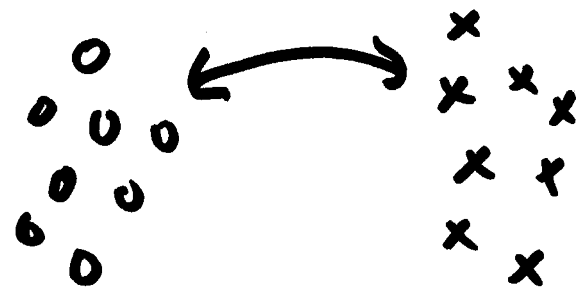
$C = r + u = 17 + 7 = 24$

8 points
3 equations per point
24 equations

7-parameter transformation

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \lambda M \begin{pmatrix}X\\Y\\Z\end{pmatrix} + \begin{pmatrix}t_x\\t_y\\t_z\end{pmatrix} \checkmark$$

$$= \lambda M \begin{pmatrix}X-t_x\\Y-t_y\\Z-t_z\end{pmatrix}$$



$n_p = 8$

$n = n_p \times 6 = 8 \times 6 = 48$

$n_0 = 7 +$ all ~~points~~ in System 1

all obs. in System 1

$n_0 \quad 7 + 8 \times 3 \; (24) = 31$

$r = 17$

$$\frac{\partial F_{1i}}{\partial X_i} = -\lambda m_{11} \qquad \frac{\partial F_{1i}}{\partial Y_i} = -\lambda m_{12} \qquad \frac{\partial F_{1i}}{\partial Z_i} = -\lambda m_{13}$$

$$\frac{\partial F_{2i}}{\partial X_i} = -\lambda m_{21} \qquad \frac{\partial F_{2i}}{\partial Y_i} = -\lambda m_{22} \qquad \frac{\partial F_{2i}}{\partial Z_i} = -\lambda m_{23}$$

$$\frac{\partial F_{3i}}{\partial X_i} = -\lambda m_{31} \qquad \frac{\partial F_{3i}}{\partial Y_i} = -\lambda m_{32} \qquad \frac{\partial F_{3i}}{\partial Z_i} = -\lambda m_{33}$$

$$\underset{c,n}{A} = \underset{24,48}{A}$$

$$\begin{array}{cc}
x_1\, y_1\, z_1 & X_1\, Y_1\, Z_1 & & & & & & \\
\left[\begin{array}{cccccccc}
I & -\lambda M & & & & & & \\
 & & I & -\lambda M & & & \phi & \\
 & & & & I & -\lambda M & & \\
 & \phi & & & & & \ddots & \\
 & & & & & & I & -\lambda M
\end{array}\right]
\end{array}$$

$$\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{-} - \lambda M \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{-} - \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{\partial F}{\partial l} = A$$

$$\frac{\partial F_{1i}}{\partial x_i} = 1 \qquad \frac{\partial F_{1i}}{\partial y_i} = 0 \qquad \frac{\partial F_{1i}}{\partial z_i} = 0$$

$$\frac{\partial F_{2i}}{\partial x_i} = 0 \qquad \frac{\partial F_{2i}}{\partial y_i} = 1 \qquad \frac{\partial F_{2i}}{\partial z_i} = 0$$

$$\frac{\partial F_{3i}}{\partial x_i} = 0 \qquad \frac{\partial F_{3i}}{\partial y_i} = 0 \qquad \frac{\partial F_{3i}}{\partial z_i} = 1$$

$$\underset{c,u}{B} \quad , \quad \underset{24,7}{B}$$

$$\begin{pmatrix} \partial F_1/\partial \lambda \\ \partial F_2/\partial \lambda \\ \partial F_3/\partial \lambda \end{pmatrix} = -M \begin{pmatrix} X_i \\ Y_i \\ Z_i \end{pmatrix}$$

next ω, φ, K

$$\begin{pmatrix} \partial F_1/\partial \omega \\ \partial F_2/\partial \omega \\ \partial F_3/\partial \omega \end{pmatrix} = -\lambda M_K M_\phi \frac{\partial M_\omega}{\partial \omega} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

etc. for ϕ, K

$$\begin{pmatrix} \frac{\partial F_1}{\partial t_x} \\ \frac{\partial F_2}{\partial t_x} \\ \frac{\partial F_3}{\partial t_x} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{\partial F_1}{\partial t_y} \\ \frac{\partial F_2}{\partial t_y} \\ \frac{\partial F_3}{\partial t_y} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial F_1}{\partial t_z} \\ \frac{\partial F_2}{\partial t_z} \\ \frac{\partial F_3}{\partial t_z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

	λ	ω	φ	κ	t_x	t_y	t_z
p_1^*	$-M\begin{pmatrix}X\\Y\\Z\end{pmatrix}$	$-\lambda M_\kappa M_\varphi \frac{\partial M_\omega}{\partial \omega}\begin{pmatrix}X\\Y\\Z\end{pmatrix}$	"	"	$\begin{matrix}-1\\0\\0\end{matrix}$	$\begin{matrix}0\\-1\\0\end{matrix}$	$\begin{matrix}0\\0\\-1\end{matrix}$
p_2^*							

B-matrix

$$f = -F^0 - A(\ell - \ell^0)$$

$$\begin{array}{ccccccc} f_{point_i} & = & -F^0_{point_i} & - & A_i & \times & \begin{pmatrix} x_1 - x_1^0 \\ y_1 - y_1^0 \\ z_i - z_1^0 \\ X_1 - X_1^0 \\ Y_1 - Y_1^0 \\ Z_1 - Z_1^0 \end{pmatrix} \\ (3,1) & & 3,1 & & 3,6 & & 6,1 \end{array}$$

W, A, B, f

full N.E.

$$\begin{pmatrix} -W & A^T & 0 \\ A & 0 & B \\ 0 & B^T & 0 \end{pmatrix} \begin{pmatrix} V \\ K \\ \Delta \end{pmatrix} = \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}$$

$X = X + \Delta$

$l^0 = l + v$

do again

---

partioned solution (vs. Full N.E.)

$Q_e = AQA^T \qquad Q = W^{-1}$

$W_e = Q_e^{-1}$

$V = QA^T K$

$$\left.\begin{aligned} N &= B^T W_e B \\ t &= B^T W_e f \end{aligned}\right\} \quad \Delta = N^{-1} t$$

$X = X + \Delta$

$l^0 = l + v$

next iteration

$K = W_e (f - B\Delta)$