$$M_\kappa M_\phi M_\omega =$$

$$\begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix} =$$

$$\begin{bmatrix} \cos\phi\cos\kappa & \cos\omega\sin\kappa + \sin\omega\sin\phi\cos\kappa & \cdots \\ -\cos\phi\sin\kappa & \cos\omega\cos\kappa - \sin\omega\sin\phi\sin\kappa & \cdots \\ \sin\phi & -\sin\omega\cos\phi & \cdots \end{bmatrix}$$

$$\begin{bmatrix} \cdots & \sin\omega\sin\kappa - \cos\omega\sin\phi\cos\kappa \\ \cdots & \sin\omega\cos\kappa + \cos\omega\sin\phi\sin\kappa \\ \cdots & \cos\omega\cos\phi \end{bmatrix}$$

[Note correction from video version ! (m_{32})]

$$F = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \lambda M \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} - \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix} = 0$$

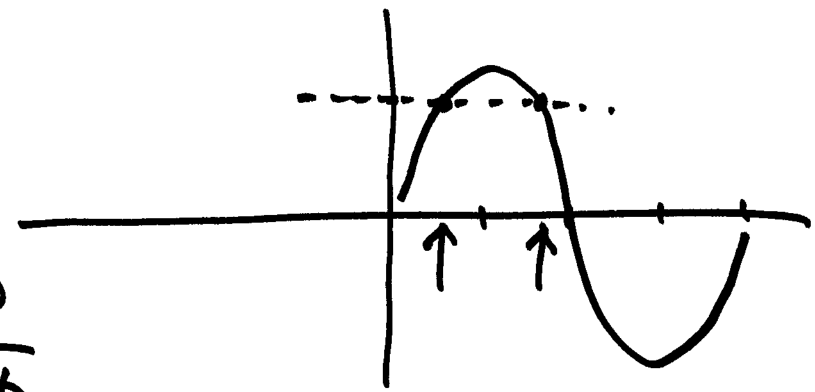
$$\frac{\partial F}{\partial \omega}, \quad \frac{\partial F}{\partial \varphi}, \quad \frac{\partial F}{\partial \kappa}$$

$$\frac{\partial F}{\partial \kappa} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \lambda \underbrace{\begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix}} M_\varphi M_\omega \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} - \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix} = 0$$

$$\frac{\partial F}{\partial \kappa} = -\lambda \begin{bmatrix} -\sin\kappa & \cos\kappa & 0 \\ -\cos\kappa & -\sin\kappa & 0 \\ 0 & 0 & 0 \end{bmatrix} M_\varphi M_\omega \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

extracting angles from matrix, ω, φ, κ

$$\phi = \sin^{-1}(m_{31})$$



$$\frac{m_{32}}{m_{33}} = \frac{-\sin\omega \cdot \cos\phi}{\cos\omega \cdot \cos\phi}$$

$$\omega = \tan^{-1}\left(-m_{32}/m_{33}\right) \qquad (-90^\circ < \phi < 90^\circ)$$

$$\frac{m_{21}}{m_{11}} = \frac{-\sin\kappa \cdot \cos\phi}{\cos\kappa \cdot \cos\phi}$$

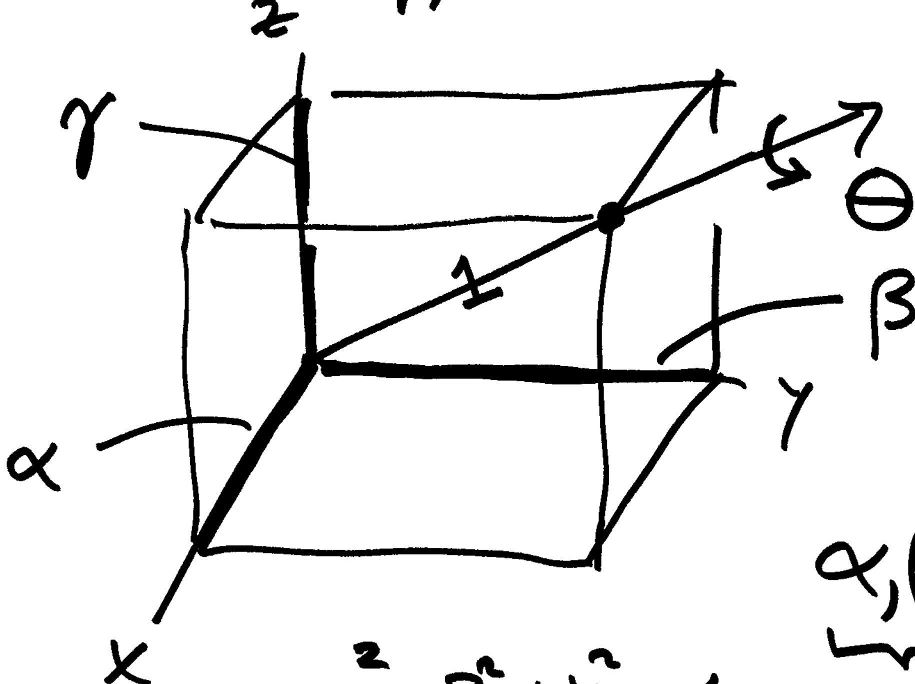
$$\kappa = \tan^{-1}\left(-m_{21}/m_{11}\right)$$

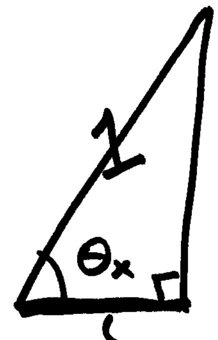
if ϕ outside range $-90^\circ \rightarrow +90^\circ$

$$\omega = \tan^{-1}\left(\frac{-m_{32}/\cos\phi}{m_{33}/\cos\phi}\right)$$

$$\kappa = \tan^{-1}\left(\frac{-m_{21}/\cos\phi}{m_{11}/\cos\phi}\right)$$

another approach to constructing 3x3 Rot. Mx





$\cos\theta_x = \alpha$

$\alpha^2+\beta^2+\gamma^2 = 1$

$\underbrace{\alpha, \beta, \gamma}, \Theta$

$$M_{3\times 3} = \begin{bmatrix} \alpha^2(1-\cos\theta)+\cos\theta & \alpha\beta(1-\cos\theta)-\gamma\sin\theta & \alpha\gamma(1-\cos\theta)+\beta\sin\theta \\ \alpha\beta(1-\cos\theta)+\gamma\sin\theta & \beta^2(1-\cos\theta)+\cos\theta & \beta\gamma(1-\cos\theta)-\alpha\sin\theta \\ \alpha\gamma(1-\cos\theta)-\beta\sin\theta & \beta\gamma(1-\cos\theta)+\alpha\sin\theta & \gamma^2(1-\cos\theta)+\cos\theta \end{bmatrix}$$

related closely to rotations represented by Quaternions.

Quaternions:

$$q = \begin{bmatrix} q_i \\ q_j \\ q_k \\ q_s \end{bmatrix}$$

unit q: $\|q\| = 1$

$$q_s = \cos\frac{\theta}{2}$$

$$\begin{pmatrix} q_i \\ q_j \\ q_k \end{pmatrix} = \sin\frac{\theta}{2} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$\cos\theta = q_s^2 - (q_i^2 + q_j^2 + q_k^2)$$

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \frac{1}{\left[q_i^2 + q_j^2 + q_k^2\right]^{1/2}} \cdot \begin{pmatrix} q_i \\ q_j \\ q_k \end{pmatrix}$$

two places see Quaternions

(1) computer graphics

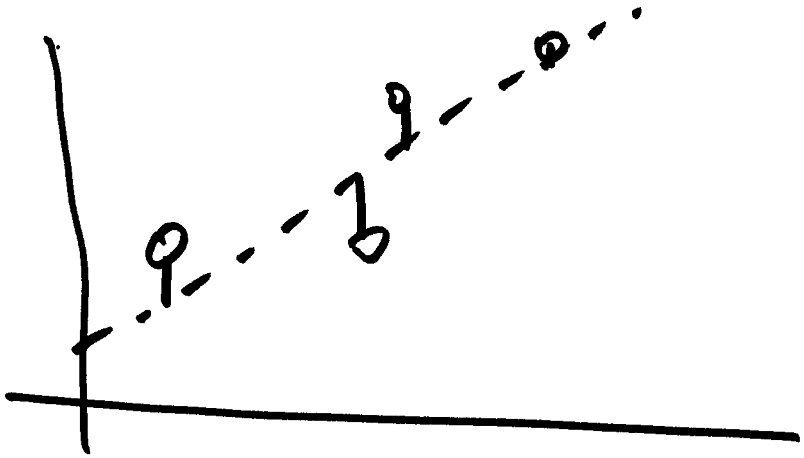
animations, interpolation of angles

(2) satellite attitude trajectory

$$\begin{bmatrix} q_i \\ q_j \\ q_k \\ q_s \end{bmatrix} \quad \Sigma = \begin{bmatrix} \sigma_{q_i}^2 & \sigma_{q_i q_j} & & \\ & \sigma_{q_j}^2 & & \\ & & \sigma_{q_k}^2 & \\ & & & \end{bmatrix}$$

constructing rotation matrix directly from Q

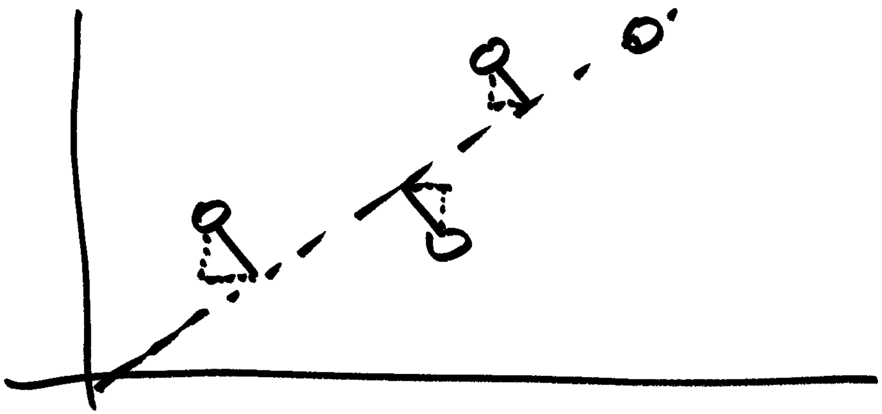
$$M = \begin{bmatrix} q_s^2 + q_i^2 - q_j^2 - q_k^2 & 2(q_j q_i - q_s q_k) & 2(q_i q_k + q_s q_j) \\ 2(q_j q_i + q_s q_k) & q_s^2 - q_i^2 + q_j^2 - q_k^2 & 2(q_j q_k - q_s q_i) \\ 2(q_i q_k - q_s q_i) & 2(q_j q_k + q_s q_i) & q_s^2 - q_i^2 - q_j^2 + q_k^2 \end{bmatrix}$$

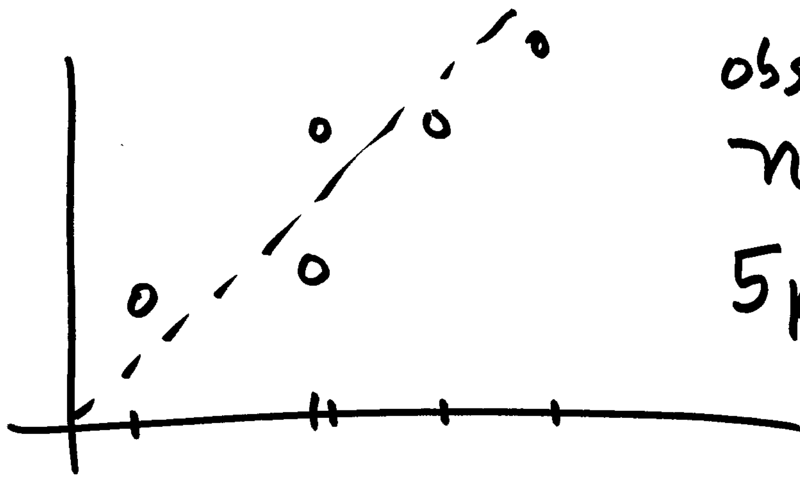


original: x constant
y observation

$\hat{y} = mx + b$

Ind. Obs.





observed in both x, y

n, n_0, r

5 pts, 2 obs/pnt

$n = 10$

n_0: 2 (for line) + 5 (one coord. component per point)

$= 7$

$r = 3$

observed only in y

$n = 5$

$n_0 = 2$

$r = 3$

original regression $\hat{y} = mx + b$

$\underline{\text{linear}}$

new GLS regression $\hat{y} = \underline{m}\,\underline{\hat{x}} + b$

$\underline{\text{non}}\ \underline{\text{linear}}$

Derivation (Ch. 9) GLS

n, n_0, r

$c = r + u$

u : chosen number of parameter

$F_1(\hat{l}, x) = 0$

$F_2(\hat{l}, x) = 0$

$\vdots$

$F_c(\hat{l}, x) = 0$

<u>non-linear</u>

taylor series approximation

$$F(\hat{l}, x) \approx F^\circ(l^\circ, x^\circ) + \underbrace{\frac{\partial F}{\partial l}}_{A} \cdot \Delta l + \underbrace{\frac{\partial F}{\partial x}}_{B} \Delta x = 0$$

$$l + v = l^0 + \Delta l$$

$$\Delta l = \underbrace{l - l^0 + v}$$

$$F^0(l^0, x^0) + A\,\Delta l + B\,\Delta x = 0$$

$$F^0(l^0, x^0) + \underbrace{A\,(l - l^0 + v)}_{A\,(l-l^0) + Av} + B\,\Delta x = 0$$

$$Av + B\Delta = \underbrace{-F^0(l^0, x^0) - A\,(l - l^0)}$$

$$\boxed{Av + B\Delta = f}$$

$$\phi' = v^T W v - 2k^T (Av + Bo - f)$$

$$\frac{\partial \phi'}{\partial v} = \underset{(1,n)}{2v^T}\underset{(n,n)}{W} - 2k^T A = \underset{(1,n)}{0}$$

$a^T b = sc.$
$b^T a$

$$\frac{\partial \phi'}{\partial o} = -\underset{1,c}{2k^T}\underset{c,u}{B} = \underset{1,u}{0}$$

$$-2(Av+Bo-f)^T k$$

$$\frac{\partial \phi'}{\partial k} = -2(Av+Bo-f)^T = 0$$

divide by 2

$$v^T W - k^T A = 0$$
$$-k^T B = 0$$
$$-(Av+Bo-f)^T = 0$$

$$A = \frac{\partial F}{\partial l}, \quad B = \frac{\partial F}{\partial x}, \quad f = -F^0(l^0, x^0) - A(l - l^0)$$

maintain 2 obs. vectors

l: original obs.

l^0: current estimate of $\vec{l}$

minimize $\underbrace{v^T W v - 2K^T(Av + B\delta - f)}_{\phi'}$

K: vector of Lagrange Multipliers

$\partial \phi' / \partial v = 0$

$\partial \phi' / \partial \delta = 0$

$\partial \phi' / \partial K = 0$

transpose

$$Wv - A^T k = 0$$

$$-B^T k = 0$$

$$-(Av + B\Delta - f) = 0$$

$\times -1$

$$-Wv + A^T k = 0$$

$$Av + B\Delta = f$$

$$B^T k = 0$$

$$\begin{bmatrix} -W & A^T & 0 \\ A & 0 & B \\ 0 & B^T & 0 \end{bmatrix} \begin{bmatrix} v \\ k \\ \Delta \end{bmatrix} = \begin{bmatrix} 0 \\ f \\ 0 \end{bmatrix}$$

full normal equations

GLS