Error Propogation

$$Y = \underset{\substack{\uparrow \\ \text{matrix of} \\ \text{constants}}}{A} X + b$$

b ← constant vector

X ← random variable Σ_{xx}

$$\boxed{\Sigma_{yy} = A \Sigma_{xx} A^T}$$

$$Y = Y^0 + J \Delta x$$ Taylor Series

Σ_{xx}

$\Sigma_{\Delta x}$

$$\boxed{\Sigma_{yy} = J \Sigma_{xx} J^T}$$

$$\begin{bmatrix} X_a \\ Y_a \\ X_b \\ Y_b \end{bmatrix} \;\&\; \Sigma_{xx} \quad \begin{bmatrix} \sigma_{X_a}^2 & \sigma_{X_aY_a} & \cdot & \cdot \\ \sigma_{X_aY_a} & \sigma_{Y_a}^2 & \cdot & \cdot \\ \sigma_{X_bX_a} & \sigma_{X_bY_a} & \sigma_{X_b}^2 & \cdot \\ \sigma_{Y_bX_a} & \sigma_{Y_bY_a} & \sigma_{Y_bX_b} & \sigma_{Y_b}^2 \end{bmatrix}$$

$$D_x = X_b - X_a$$
$$D_y = Y_b - Y_a$$

what is $\Sigma_{\binom{D_x}{D_y}}$?

$$\begin{pmatrix} D_x \\ D_y \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_a \\ Y_a \\ X_b \\ Y_b \end{pmatrix}$$

$$\Sigma_{\binom{D_x}{D_y}} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \Sigma_{xx} \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Sigma_{\begin{pmatrix}D_x\\D_y\end{pmatrix}} = \begin{pmatrix}\sigma^2_{D_x} & \sigma_{D_xD_y}\\ \sigma_{D_xD_y} & \sigma^2_{D_y}\end{pmatrix}$$

error propagation with either $\left\{\begin{array}{ll}\Sigma & \text{abs. cov. mx}\\ Q & \text{Scaled cov. mx.}\end{array}\right.$

$$\Sigma = \underline{\underline{\sigma_0^2}}\, Q$$

$$\Sigma_{xx} = \sigma_0^2 Q_{xx} \qquad \Sigma_{\hat{\ell}\hat{\ell}} = \sigma_0^2 Q_{\hat{\ell}\hat{\ell}}$$

$$\Sigma_{\Delta\Delta} = \sigma_0^2 Q_{\Delta\Delta}$$

$$\Sigma_{vv} = \sigma_0^2 Q_{vv}$$

$$\Sigma_{\ell\ell} = \sigma_0^2 Q_{\ell\ell} \quad (Q^{*} = Q_{\ell\ell})$$

$\Sigma_{xx} = \sigma_0^2 Q_{xx}$

σ_0^2 either $\begin{cases} \text{prior } \sigma_0^2 \text{ chosen "a priori"} \\ \text{post adjustment } \hat{\sigma}_0^2 \text{ estimated "a posteriori"} \end{cases}$

<u>Indirect Observation Model</u>

$Q_{\delta\delta} = N^{-1} = (B^T W B)^{-1}$
(Q_{xx})

$\Sigma_{\delta\delta} = \sigma_0^2 Q_{\delta\delta}$

$Q_{\hat{\ell}\hat{\ell}} = ?$

$$Q_{\hat{\ell}\hat{\ell}} = B(B^TWB)^{-1}B^TW \cdot Q \left[B(B^TWB)^{-1}B^TW\right]^T$$

$$WB(B^TWB)^{-1}B^T$$

$$= B(B^TWB)^{-1}\underbrace{B^TW\boxed{QW}B}\,\underbrace{(B^TWB)}^{-1}B^T$$

$$\boxed{Q_{\hat{\ell}\hat{\ell}} = BN^{-1}B^T}$$

I/0

$$\Sigma_{\hat{\ell}\hat{\ell}} = \sigma_0^2 Q_{\hat{\ell}\hat{\ell}}$$

$$Q_{vv} = \,?\qquad v = f - B\Delta$$

$$v = d - \ell - BN^{-1}B^TW(d-\ell)$$

$$v = d - \ell - BN^{-1}B^TWd + BN^{-1}B^TW\ell$$

$$v = (I - BN^{-1}B^TW)d + (BN^{-1}B^TW - I)\ell \quad \checkmark$$

form we need

$$\hat{l} = l + v, \quad v + B\Delta = f, \quad v = f - B\Delta$$

$$f = d - l$$

$$\hat{l} = l + f - B\Delta = \underline{\underline{l}} + d - \underline{\underline{l}} - B\Delta$$

$$\hat{l} = d - B\Delta \qquad \Delta = (B^TWB)^{-1}B^TW(d-l)$$

$$\hat{l} = d - B(B^TWB)^{-1}B^TW(d-l)$$

$$\hat{l} = d - B(B^TWB)^{-1}B^TWd + B(B^TWB)^{-1}B^TWl$$

$$\left.\begin{aligned}\hat{l} &= (I - B(B^TWB)^{-1}B^TW)\,d \; + \\ &\quad \underbrace{B(B^TWB)^{-1}B^TW}\cdot\underbrace{l}\end{aligned}\right\}\text{ in form we need for error prop.}$$

$$Q_{vv} = (BN^{-1}B^TW - I)\,Q\,(BN^{-1}B^TW - I)^T$$

$$= (BN^{-1}B^T\underline{WQ} - Q)(WBN^{-1}B^T - I)$$

$$= (BN^{-1}B^T - Q)(WBN^{-1}B^T - I)$$

$$BN^{-1}\underbrace{\underline{B^TWB}\underline{N}^{-1}}B^T + Q - BN^{-1}B^T - BN^{-1}B^T$$

$$Q_{vv} = Q \underbrace{+ BN^{-1}B^T - BN^{-1}B^T}- BN^{-1}B^T$$

$$\boxed{Q_{vv} = Q - BN^{-1}B^T}$$

<u>Indirect Observations</u>

$$Q_{vv} = Q - Q_{\hat{\ell}\hat{\ell}}\;, \qquad Q_{\hat{\ell}\hat{\ell}} = Q - Q_{vv}$$

Observations Only

$\hat{l} = l + v$

$Av = f$

$\underline{v = QA^T k}$

$k = W_e f$

$Av = \underline{\underline{d - Al}}$

$Q_e = AQA^T$

$W_e = Q_e^{-1}$

$Q_{vv} = ?$ $\quad v = QA^T k, \quad = QA^T W_e f$

$v = QA^T W_e (d - Al)$

$v = \underbrace{QA^T W_e d} - \underbrace{QA^T W_e A}\,\underbrace{l}$

$Q_{vv} = QA^T W_e A \cdot Q \cdot (QA^T W_e A)^T$

$A^T W_e A Q$

$$\boxed{Q_{vv} = QA^T W_e A Q}$$

$$\hat{\ell} = \ell + QA^TW_e d - QA^TW_e A \ell$$

$$\hat{\ell} = QA^TW_e \ell + (I - QA^TW_e A)\ell \leftarrow$$

$$Q_{\hat{\ell}\hat{\ell}} = (I - QA^TW_e A)\, Q\, (I - QA^TW_e A)^T$$

$$(I - A^TW_e AQ)$$

$$= (Q - QA^TW_e AQ)\,(\quad\uparrow\quad)$$

$$Q_{\hat{\ell}\hat{\ell}} = QA^TW_e AQ - QA^TW_e AQ +$$

$$QA^TW_e \underbrace{AQA^T}\underbrace{W_e} AQ$$

$$Q_{\hat{\ell}\hat{\ell}} = QA^TW_e AQ \underbrace{- QA^TW_e AQ + QA^TW_e AQ}$$

$$\boxed{Q_{\hat{\ell}\hat{\ell}} = Q - QA^T W_e A Q}$$

$$\underline{Q_{\hat{\ell}\hat{\ell}} = Q - Q_{vv}}\,, \quad \underline{Q_{vv} = Q - Q_{\hat{\ell}\hat{\ell}}}$$

<u>post adjustment statistics</u>

Global Test (are results consistent w/ prior assumptions?)

Hypothesis Test

$$H_0 : \sigma^2 = \sigma_0^2$$

true var. of unit weight

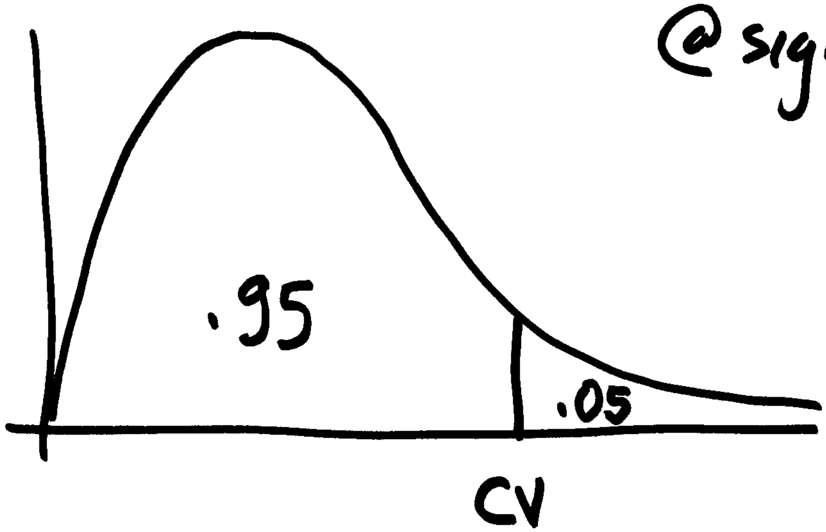
chosen value

$H_1 : \sigma^2 > \sigma_0^2$

we know from statistics, if $\underline{\sigma^2 = \sigma_0^2}$ then

$$\frac{v^T W v}{\sigma_0^2} \sim \chi^2_r$$

@ significance level of $\underline{.05}$



test statistic $\chi^* = \dfrac{v^T W v}{\sigma_0^2}$

Decision Rule:

if $\chi^* < cv$, we accept H_0

if $\chi^* > cv$, we reject H_0
accept H_1