$$E\{X\} = \mu_x$$

$$E\{(X-\mu_x)^2\} = \sigma_x^2 = \text{variance} \quad (\sigma_{xx})$$

$$E\{(X-\mu_x)(Y-\mu_y)\} = \sigma_{xy}$$

$$E\{(Y-\mu_y)(X-\mu_x) = \sigma_{yx}$$

equal

properties of Expectation

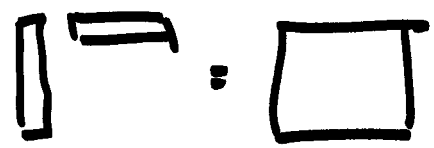
$$E\{X+Y\} = E\{X\} + E\{Y\}$$

$$E\{aX\} = a\,E\{X\}$$

$$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{random vector} \qquad E(\vec{X}) = \begin{bmatrix} E\{x_1\} \\ E\{x_2\} \\ \vdots \\ E\{x_n\} \end{bmatrix} = \begin{bmatrix} \mu_{x_1} \\ \mu_{x_2} \\ \vdots \\ \mu_{x_n} \end{bmatrix}$$

DEFN: covariance matrix

$$\underset{n,n}{\Sigma_{xx}} = E\{\underset{n,1}{(\vec{X}-\vec{\mu}_x)}\underset{1,n}{(\vec{x}-\vec{\mu}_x)^T}\}$$



$$E\left\{\begin{bmatrix} x_1-\mu_{x_1} \\ x_2-\mu_{x_2} \\ \vdots \\ x_n-\mu_{x_n} \end{bmatrix}\begin{bmatrix} x_1-\mu_{x_1} & x_2-\mu_{x_2} & \cdots & x_n-\mu_{x_n} \end{bmatrix}\right\}$$

$$E\left\{\begin{bmatrix} (x_1-\mu_{x_1})(x_1-\mu_{x_1}) & (x_1-\mu_{x_1})(x_2-\mu_{x_2}) & (x_1-\mu_{x_1})(x_n-\mu_{x_n}) \\ (x_2-\mu_{x_2})(x_1-\mu_{x_1}) & (x_2-\mu_{x_2})(x_2-\mu_{x_2}) & \vdots \\ \vdots & \ddots & \vdots \\ (x_n-\mu_{x_n})(x_1-\mu_{x_1}) & \cdots & (x_n-\mu_{x_n})(x_n-\mu_{x_n}) \end{bmatrix}\right\}$$

bring expectation inside,

$$\begin{bmatrix} E\{(x_1-\mu_{x_1})(x_1-\mu_{x_1})\} & E\{(x_1-\mu_{x_1})(x_2-\mu_{x_2}) \\ E\{(x_2-\mu_{x_2})(x_1-\mu_{x_1})\} & \cdot \end{bmatrix}$$

$$\begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1x_2} & & \sigma_{x_1x_n} \\ \sigma_{x_1x_2} & \sigma_{x_2}^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ \sigma_{x_nx_1} & \cdots & & \sigma_{x_n}^2 \end{bmatrix} = \Sigma_{xx}$$

variance/covariance matrix

covariance matrix

$$\sigma_{x_1x_2} = r_{12}\,\sigma_{x_1}\sigma_{x_2}$$

r_{ij} : correlation coefficient

$$r_{12} = \frac{\sigma_{x_1x_2}}{\sigma_{x_1}\sigma_{x_2}}$$

$$r_{ij} : -1 \rightarrow +1$$

covariance matrix

- symmetric
- positive definite $\quad x^T \Sigma_{xx} x > 0$

use <u>covariance</u> <u>matrix</u> to describe <u>precision</u> of R.V.

$$l,\ \hat{l},\ v,\ x,\ \Delta$$

$$\begin{matrix} l \\ \Sigma_{ll} \end{matrix} \longrightarrow \boxed{LS} \longrightarrow \begin{matrix} x \\ \Sigma_{xx} \end{matrix}$$

if diagonal $\Sigma_{xx} = \begin{bmatrix} \sigma_{x_1}^2 & & & 0 \\ & \sigma_{x_2}^2 & & \\ & & \ddots & \\ 0 & & & \sigma_{x_n}^2 \end{bmatrix}$

$\Rightarrow$ uncorrelated

$$\Sigma_{xx}^{-1} = \begin{bmatrix} 1/\sigma_{x_1}^2 & & & 0 \\ & 1/\sigma_{x_2}^2 & & \\ & & \ddots & \\ 0 & & & 1/\sigma_{x_n}^2 \end{bmatrix}$$

$$\sigma_0^2 \Sigma_{xx}^{-1} = \begin{bmatrix} \sigma_0^2/\sigma_{x_1}^2 & & & 0 \\ & \sigma_0^2/\sigma_{x_2}^2 & & \\ & & \ddots & \\ 0 & & & \sigma_0^2/\sigma_{x_n}^2 \end{bmatrix} \Rightarrow W_{xx}$$

$$\sigma_0^2 \Sigma_x^{-1} = W_x \qquad \begin{pmatrix} W_x = W_{xx} \\ \Sigma_x = \Sigma_{xx} \end{pmatrix}$$

$$\frac{1}{\sigma_0^2} \Sigma_x = Q_x$$

$$\boxed{\Sigma_x = \sigma_0^2 Q_x}$$

Q: cofactor matrix

"absolute" covariance

"relative" covariance

X, Σ_{xx} $\qquad$ $Y = AX$ $\qquad$ what is $\underline{\underline{\Sigma_{yy}}}$

A ↑ constant matrix

$$E\{Y\} = E\{AX\} = A\,E\{X\}$$

$$\mu_y = A\,\mu_x$$

$$\Sigma_{yy} = E\{(\underbrace{y}_{Ax} - \underbrace{\mu_y}_{A\mu_x})(y - \mu_y)^T\}$$ defn of cov. mx.

$$\Sigma_{yy} = E\left\{(Ax - A\mu_x)(Ax - A\mu_x)^T\right\} =$$

$$E\left\{(Ax - A\mu_x)(x^T A^T - \mu_x^T A^T)\right\} =$$

$$E\left\{A(x - \mu_x)\underbrace{(x^T - \mu_x^T)}_{(x-\mu_x)^T} A^T\right\} =$$

$$= A \underbrace{E\left\{(x-\mu_x)(x-\mu_x)^T\right\}}_{\Sigma_{xx}} A^T$$

$$\boxed{\Sigma_{yy} = A \Sigma_{xx} A^T}$$

diagonal

$X = M\ell$

$y = a_1 x_1 + a_2 x_2$, $\Sigma_{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{pmatrix} \sigma_{x_1}^2 & 0 \\ 0 & \sigma_{x_2}^2 \end{pmatrix}$

$y = (a_1 \; a_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\sigma_y^2 = ?$

$$\sigma_y^2 = (a_1 \; a_2) \begin{pmatrix} \sigma_{x_1}^2 & 0 \\ 0 & \sigma_{x_2}^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\sigma_y^2 = a_1^2 \sigma_{x_1}^2 + a_2^2 \sigma_{x_2}^2$$

X's are uncorrelated

$$y = a_1 x_1 + a_2 x_2 \, , \qquad \Sigma_{xx} = \begin{pmatrix} \sigma_{x_1}^2 & \sigma_{x_1 x_2} \\ \sigma_{x_1 x_2} & \sigma_{x_2}^2 \end{pmatrix}$$

$$y = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\sigma_y^2 = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} \sigma_{x_1}^2 & \sigma_{x_1 x_2} \\ \sigma_{x_1 x_2} & \sigma_{x_2}^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\underline{\sigma_y^2 = a_1^2 \sigma_{x_1}^2 + \underbrace{2 a_1 a_2 \sigma_{x_1 x_2}}_{} + a_2^2 \sigma_{x_2}^2}$$

$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, $\Sigma_{xx} = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$, $\begin{array}{l} Y_1 = X_1 + 2X_2 \\ Y_2 = 2X_1 + X_2 \end{array}$,

$Z = Y_1 + 2Y_2$, $\sigma_Z^2 = ?$

<u>1-step</u> (substitution)

$$Z = X_1 + 2X_2 + 2(2X_1 + X_2)$$

$$Z = 5X_1 + 4X_2$$

$$Z = \begin{pmatrix} 5 & 4 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$\sigma_Z^2 = \begin{pmatrix} 5 & 4 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix} = 204$$

$$\sigma_Z = \sqrt{204}$$

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$$\left.\begin{aligned} Y_1 &= X_1 + 2X_2 \\ Y_2 &= 2X_1 + X_2 \end{aligned}\right\} \quad \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \underline{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad \Sigma_x = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$$

$$\Sigma_{YY} = A \Sigma_x A^T = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 24 & 21 \\ 21 & 24 \end{pmatrix}$$

$$Z = Y_1 + 2Y_2 = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

$$\sigma_Z^2 = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 24 & 21 \\ 21 & 24 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 204$$

$$\sigma_Z = \sqrt{204}$$

I/O : indirect observations

$$\boxed{Y = Ax,\ \Sigma_{xx}}$$

$$Y = Ax + b$$

$$\Delta = \underbrace{(B^T W B)^{-1} B^T W}\, f$$

$$\underbrace{f} = d - l = \underbrace{d - Il}, \quad f = d - Il$$

can do error propagation with abs Σ

rel Q

$$\boxed{Y = Ax,\quad Q_{yy} = A Q_{xx} A^T}$$

have W_{ll}

$$\underline{\underline{Q_{ff}}} = -I Q_{ll} -I^T = Q_{ll}$$

$$Q_{\Delta\Delta} = (B^T W B)^{-1} B^T W Q_{ll} \left[(B^T W B)^{-1} B^T W \right]^T$$

$$W B (B^T W B)^{-1}$$

$$Q_{\Delta\Delta} = (B^T W B)^{-1} \underbrace{\underbrace{B^T \underbrace{W Q}_{I} W B}\,\underbrace{(B^T W B)^{-1}}}_{I}$$

$$Q_{\Delta\Delta} = (B^T W B)^{-1}, \quad \boxed{Q_{\Delta\Delta} = N^{-1}}$$

$$\Delta = \underbrace{N}^{-1} t$$

$$Q_{\Delta\Delta} = N^{-1}$$

$$\Sigma_{\Delta\Delta} = \sigma_0^2 Q_{\Delta\Delta}$$