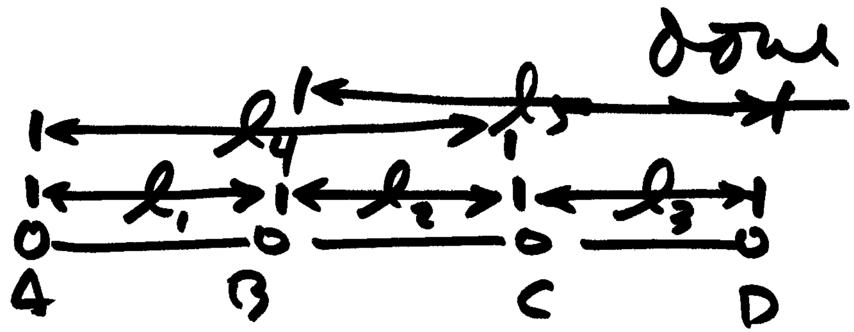
solution of observations only LS problem
– substitution method

1. analyze problem n, n_0, r
2. write r condition eqn's among $\hat{l}_i$ which express math model
3. put in form v's on left, const on right
4. Separate n v_i's into 2 groups
   n_0 to keep (must define model)
   r to eliminate
5. solve for r v's in terms of the n_0 v's
6. plug expressions in $\Phi = \sum v_i^2$ with only n_0 v's
7. differentiate w.r.t. n_0 v's, set $= 0$
8. solve NE for v's

9. go back to 5 and get other V's

10. $\hat{l}_i = l_i + v$



$n = 5$
$n_0 = 3$
$\overline{\quad}$
$r = 2$

	l's
l_1	100
l_2	100
l_3	100.08
l_4	200.04
l_5	200

$$\hat{l}_1 + \hat{l}_2 = \hat{l}_4 \qquad l_1 + V_1 + l_2 + V_2 = l_4 + V_4$$

$$\hat{l}_2 + \hat{l}_3 = \hat{l}_5 \qquad l_2 + V_2 + l_3 + V_3 = l_5 + V_5$$

$$100 + V_1 + 100 + V_2 - 200.04 - V_4 = 0$$

$$100 + V_2 + 100.08 + V_3 - 200 - V_5 = 0$$

$$V_1 + V_2 - V_4 = -100 - 100 + 200.04$$

$$V_2 + V_3 - V_5 = -100 - 100.08 + 200$$

$$V_1 + V_2 - V_4 = .04$$
$$V_2 + V_3 - V_5 = -.08$$

partition: keep $\vec{l}_1, \vec{l}_2, \vec{l}_3$ (n_0)
eliminate $\vec{l}_4, \vec{l}_5$ (r)

$$V_4 = V_1 + V_2 - .04$$
$$V_5 = V_2 + V_3 + .08$$

$$\Phi = V_1^2 + V_2^2 + V_3^2 + (V_1 + V_2 - .04)^2 + (V_2 + V_3 + .08)^2$$
$$\partial\Phi/\partial V_1 = \cancel{2}V_1 + \cancel{2}(V_1 + V_2 - .04) = 0$$
$$\partial\Phi/\partial V_2 = \cancel{2}V_2 + \cancel{2}(V_1 + V_2 - .04) + \cancel{2}(V_2 + V_3 + .08) = 0$$
$$\partial\Phi/\partial V_3 = \cancel{2}V^3 + \cancel{2}(V_2 + V_3 + .08) = 0$$

$$\left.\begin{aligned} 2V_1 + V_2 &= .04 \\ V_1 + 3V_2 + V_3 &= -.04 \\ V_2 + 2V_3 &= -.08 \end{aligned}\right\} \quad \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} .04 \\ -.04 \\ -.08 \end{bmatrix}$$

$$N \quad v = t$$

$$\underbrace{v = N^{-1} t}_{}$$

solve linear system using matlab, mathcad, etc.

$V_1 = .025$

$V_2 = -.010$

$V_3 = -.035$

$V_4 = V_1 + V_2 - .04 = -.025$

$V_5 = V_2 + V_3 + .08 = .035$

hand solution of obs. only using substitution

instead of substitution $\Rightarrow$ <u>LM</u> lagrange multipliers easier to <u>automate</u>!

Solution of obs. only using Lagrange Multipliers

1. analyze n, n_0, r
2. write r cond. among $\hat{l}_i$, express math model
3. write augmented obj. function

$$\Phi' = \sum v_i^2 + \underset{\uparrow 2}{\lambda_1}(f_1(v)) + \underset{\uparrow 2}{\lambda_2}(f_2(v)) + \cdots + \underset{\uparrow 2}{\lambda_c}(f_c(v))$$

$*$ $\qquad c = r$

4. differentiate ~~ox~~ Φ' w.r.t. $n+r$ unknowns v's + λ's, $(= 0)$
5. solve system for v's, λ's
6. $\hat{l} = l_0 + v$

done.

$*$ factor of 2 added to Φ' is a matter of choice. it makes the hand comps easier. no effect on v_i, $\hat{l}_i$, etc.

$$\Phi' = V_1^2 + V_2^2 + V_3^2 + V_4^2 + V_5^2 + \lambda_1(V_1 + V_2 - V_4 - .04) + \lambda_2(V_2 + V_3 - V_5 + .08)$$

$$\partial\phi'/\partial V_1 = 2V_1 + \lambda_1 = 0$$

$$\partial\phi'/\partial V_2 = 2V_2 + \lambda_1 + \lambda_2 = 0$$

$$\partial\phi'/\partial V_3 = 2V_3 + \lambda_2 = 0$$

$$\partial\phi'/\partial V_4 = 2V_4 - \lambda_1 = 0$$

$$\partial\phi'/\partial V_5 = 2V_5 - \lambda_2 = 0$$

$$\partial\phi'/\partial \lambda_1 = V_1 + V_2 - V_4 = .04$$

$$\partial\phi'/\partial \lambda_2 = V_2 + V_3 - V_5 = -.08$$

$n + r$

$5 + 2$

7 eq.

7 unk.

differentiate Φ' w.r.t. all unknowns.

$$
\begin{array}{c}
\begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & \lambda_1 & \lambda_2 \end{array} \\
\left[\begin{array}{ccccc:cc}
2 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 2 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 2 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 2 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & -1 \\
\hdashline
1 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & -1 & 0 & 0
\end{array}\right]
\end{array}
\left[\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ \hdashline \lambda_1 \\ \lambda_2 \end{array}\right]
=
\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hdashline .04 \\ -.08 \end{array}\right]
$$

full normal equations with partition

Structure

$$
\left[\begin{array}{c} .025 \\ -.010 \\ -.035 \\ -.025 \\ .035 \\ \hdashline -.05 \\ .07 \end{array}\right]
\begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right\} v_s' \\ \left.\begin{array}{c} \\ \\ \end{array}\right\} \lambda \end{array}
$$

alternative for LM / hand solution

solve for V's in terms of λ's

plug λ's into ~~the~~ condition equations

r eqn's r unknowns

$$2v_1 + \lambda_1 = 0 \qquad v_1 = -\tfrac{1}{2}\lambda_1$$

$$2v_2 + \lambda_1 + \lambda_2 = 0 \qquad v_2 = -\tfrac{1}{2}\lambda_1 - \tfrac{1}{2}\lambda_2$$

$$2v_3 + \lambda_2 = 0 \qquad v_3 = -\tfrac{1}{2}\lambda_2$$

$$2v_4 - \lambda_1 = 0 \qquad v_4 = \tfrac{1}{2}\lambda_1$$

$$2v_5 - \lambda_2 = 0 \qquad v_5 = \tfrac{1}{2}\lambda_2$$

substitute into cond. eqn.

$$V_1 + V_2 - V_4 = .04$$
$$V_2 + V_3 - V_5 = -.08$$

$$-.5\lambda_1 - .5\lambda_1 - .5\lambda_2 - .5\lambda_1 = .04$$
$$-.5\lambda_1 - .5\lambda_2 - .5\lambda_2 - .5\lambda_2 = -.08$$

$$-\lambda_1 - \lambda_1 - \lambda_2 - \lambda_1 = .08$$
$$-\lambda_1 - \lambda_2 - \lambda_2 - \lambda_2 = -.16$$

$$\left.\begin{aligned} -3\lambda_1 - \lambda_2 &= .08 \\ -\lambda_1 - 3\lambda_2 &= -.16 \end{aligned}\right\} \quad \begin{bmatrix} -3 & -1 \\ -1 & -3 \end{bmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} .08 \\ -.16 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -.05 \\ .07 \end{pmatrix}$$

next, plug λ's into expressions for V's (pg. 4-8)

conclusions

1. ind. obs. $n_o \times n_o$ $\underline{3 \times 3}$

2. obs only substitution $n_o \times n_o$ $\underline{3 \times 3}$

3. obs. only / LM
   - full $n{+}r \times n{+}r$ $\underline{7 \times 7}$
   - part $r \times r$ $\underline{2 \times 2}$

LS method

dimension of linear system to solve